FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-second meeting of the Society was held in New York City on Saturday, February 25, 1910. The following thirty-eight members attended the two sessions:

Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor E. W. Brown, Mr. R. D. Carmichael, Professor F. N. Cole, Professor J. L. Coolidge, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Professor W. B. Fite, Dr. D. C. Gillespie, Professor C. C. Grove, Professor H. E. Hawkes, Mr. Robert Henderson, Professor Percy Hodge, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Dr. N. J. Lennes, Mr. P. H. Linehan, Professor James Maclay, Mr. A. R. Maxson, Dr. E. J. Miles, Dr. H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Mr. W. M. Smith, Professor Elijah Swift, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Professor H. S. White, Miss E. C. Williams.

The President of the Society, Professor H. B. Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. Elizabeth R. Bennett, University of Nebraska; Mr. Daniel Buchanan, University of Chicago; Dr. H. B. Curtis, Columbia University; Mr. L. L. Dines, University of Chicago; Professor C. R. MacInnes, Princeton University; Professor Eva S. Maglott, Ohio Northern University; Mr. R. E. Root, University of Chicago; Professor Sarah E. Smith, Mount Holyoke College. Six applications for membership in the Society were received.

The following papers were presented at this meeting:

(1) Dr. E. J. MILES: "Some properties of space curves minimizing a definite integral with discontinuous integrand."

(2) Dr. N. J. LENNES: "A necessary and sufficient condition for the uniform convergence of a certain class of infinite series."

(3) Dr. N. J. LENNES: "Duality in projective geometry."

(4) Professor G. A. MILLER: "The number of abelian subgroups in the possible groups of order 2^m."

(5) Professor C. N. MOORE: "On the uniform convergence of the developments in Bessel functions."

(6) Professor G. D. BIRKHOFF: "A direct method for the summation of developments in Lamé's functions and of allied developments."

(7) Professor EDWARD KASNER: "Equitangentials in space."

(8) Professor EDWARD KASNER: "Conformal and equilong invariants of horn angles."

(9) Professor J. A. EIESLAND: "On a contact transformation in physics."

(10) Dr. D. C. GILLESPIE: "Definite integrals containing a parameter."

(11) Professor JOSEPH BOWDEN: "The Russian peasant method of multiplication."

(12) Dr. N. J. LENNES: "A direct proof of the theorem that the number of terms in the expansion of an infinite determinant is of the same potency as the continuum."

(13) Professor HARRIS HANCOCK: "On algebraic equations that are connected with the cyclotomic equations and the realms of rationality which they determine" (preliminary communication).

(14) Professor W. B. FITE: "Irreducible homogeneous linear groups of order p^m and of degree p or p^2."

In the absence of the authors, the papers of Professors Miller, Moore, Eiesland and Hancock were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In dealing with integrals of the form

$$J = \int F(x, y, z, x', y', z')dt$$

it is customary to assume that F is a continuous function of its six arguments. In Dr. Miles's paper some properties of a minimizing curve

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

are studied when the integrand has a finite discontinuity for points (x, y, z) on a given surface. Three of the necessary conditions which such a curve must satisfy in order to minimize the integral are derived from the known conditions in the ordinary space problem of the calculus of variations. A fourth con-

dition, corresponding to the Jacobi condition, is then derived and it is shown that these conditions, with slight alterations, are also sufficient.

2. In this paper Dr. Lennes gives a necessary and sufficient condition for the uniform convergence over a certain interval of the series

$$\sum_{n=0}^{\infty} U_n(x) = f(x)$$

provided each term of the series and also $f(x)$ are continuous functions in the given interval.

3. In this paper Dr. Lennes gives two sets of fundamental propositions, each sufficient to characterize general projective space. The first set is stated in terms of the abstract symbols "point" and "plane" and two undefined relations "point on plane" and "plane on point." The second set is stated in terms of point, line, and plane, and six undefined relations point on line, line on point, etc. Each set of propositions is dual with respect to point and plane.

4. The theorem that every group G of order p^m, p being an arbitrary prime number, contains an abelian subgroup of order p^α whenever $m > \frac{1}{2}\alpha(\alpha-1)$ was proved in the *Messenger of Mathematics*, volume 27 (1897–98), page 120. In a later number of the same journal, volume 36 (1906–07), page 79, it was observed that the given theorem can be stated more completely by adding that G contains an *invariant* abelian subgroup of order p^α whenever $m > \frac{1}{2}\alpha(\alpha-1)$. Moreover, it was observed in this later article that in the very special case where $p=2$ and $\alpha=4$ it is possible to extend the theorem, since every group of order 64 contains an abelian invariant subgroup of order 16. In the present paper Professor Miller proves that the theorem in question can be extended for all values of $\alpha > 3$ when $p=2$. The following theorems are established:

Every group of order 2^m contains an invariant abelian subgroup of order 2^β whenever $m \geqq \frac{1}{2}\beta(\beta-1)$. The number of the abelian subgroups of order p^α in any group of order p^m is of the form $kp+1$ whenever it is not zero.

5. This paper supplements an earlier one* presented to the Society by Professor Moore. It adds to the results there

* Abstract in BULLETIN, vol. 16 (1909–10), p. 173.

obtained for developments in terms of Bessel functions of order zero, and extends the discussion to Bessel functions of order ν, where $\nu > 0$. The combined papers have been offered for publication to the *Transactions*.

6. A pair of ordinary linear differential equations of the second order

$$L(u) \equiv \frac{d^2u}{dx^2} + \phi(x)u + [\lambda a(x) + \mu b(x)]u = 0,$$

$$M(v) \equiv \frac{d^2v}{dy^2} + \psi(y)v + [\lambda c(y) + \mu d(y)]v = 0,$$

$$[a(x)d(y) - b(x)c(y) > 0],$$

will both have solutions $u_i(x)$, $v_i(y)$ which vanish for $x = m_1, m_2$, $y = n_1, n_2$ respectively for certain sets λ_i, μ_i of values of the parameters λ, μ. These solutions give rise to the formal development of an arbitrary function $f(x, y)$ of two variables,

$$f(x, y) \sim \sum c_i u_i(x) v_i(y),$$

including as a special case developments in Lamé's products. Professor Birkhoff treats the question whether this series represents the function by the use of a contour integral

$$\int_C \int_D \int_{m_1}^{m_2} \int_{n_1}^{n_2} f(\xi, \eta) G(x, \xi) H(y, \eta) d\xi\, d\eta\, d\lambda\, d\mu,$$

where G and H are the one-dimensional Green's functions belonging to the first and second of the given equations respectively, under the given boundary conditions, and C and D are properly chosen contours in the λ and μ planes respectively. This integral yields an explicit expression for the sum of any number of terms of the given series, and, by means of a study of the asymptotic character of the solution of the given equations for λ and μ large in absolute value, this series is shown to tend toward $f(x, y)$, where $f(x, y)$ is continuous, under suitable restrictions. The limiting value at a point of discontinuity is also discussed.

The method outlined above is of decidedly greater power than that of integral equations which Hilbert has recently employed, and is to be characterized as an extension of the method of residues.

7. Given any congruence of curves in space, we may obtain ∞^1 equitangential congruences by a construction analogous to that employed by Scheffers in the case of plane systems. Professor Kasner studies the ∞^3 curves thus generated. The simplest result is that for the ∞^1 curves having a common tangent line there is a homographic correspondence between the point of contact and the osculating plane. The theory of isogonals, it may be remarked, differs radically from the plane case since the number of such trajectories for a space congruence is ∞^∞.

8. In the general case of horn angles formed by two curves having contact of the kth order, Professor Kasner shows that there exists a unique conformal invariant I_k and a unique equilong invariant J_k. The necessary and sufficient condition for equivalence in each group is that the two angles have the same order of contact and equal invariants.

In the simplest case, contact of the first order, the invariants are

$$I=\left(\frac{d\gamma_1}{ds_1}-\frac{d\gamma_2}{ds_2}\right)/(\gamma_1-\gamma_2)^2, \quad J=\left(\frac{dr_1}{d\theta_1}-\frac{dr_2}{d\theta_2}\right)/(r_1-r_2)^2,$$

where θ denotes inclination, s arc, γ curvature, and r radius of curvature, the subscripts 1 and 2 referring to the two curves forming the angle. If the values of γ and r in terms of θ and s are inserted, the two expressions differ only by the interchange of the letters θ and s.

All contact transformations turn horn angles into horn angles of the same order of contact. Those which preserve the value of I (or J) constitute the conformal (or equilong) group.

In a previous paper the author showed that ordinary curvilinear angles have under the conformal group no invariant beyond the magnitude of the angle, except when that magnitude is commensurable to 2π. Under the equilong group, the figure formed by two curves with a common tangent has the length of that tangent for its sole invariant.

9. Professor Eiesland's paper is in abstract as follows: A contact transformation

$$\delta x=\frac{\partial w}{\partial p}\delta t, \quad \delta y=\frac{\partial w}{\partial q}\delta t, \quad \delta z=\left(p\frac{\partial w}{\partial p}+q\frac{\partial w}{\partial q}-w\right)\delta t,$$

$$\delta p=0, \quad \delta q=0$$

whose characteristic function w is a function of p and q only will represent a wave motion in a non-isotropic body, the velocity depending only on the direction of the motion. A surface element is transformed parallel to itself, while a point is transformed into a surface

$$F\left(\frac{x_1-x}{t}, \frac{y_1-y}{t}, \frac{z_1-z}{t}\right)=0.$$

The velocity v, normal to the wave, is expressed by the relation

$$v=\frac{w}{\sqrt{1+p^2+q^2}}.$$

In the case of wave motion in a biaxial crystal v is defined by the equation

$$(1+p^2+q^2)v^4-[(b^2+c^2)p^2+(a^2+c^2)q^2+(a^2+b^2)]v^2 + b^2c^2p^2+a^2c^2q^2+a^2b^2=0,$$

that is, w is defined by the equation

$$w^4-[(b^2+c^2)p^2+(a^2+c^2)q^2+a^2+b^2]w^2 + (b^2c^2p^2+a^2c^2q^2+a^2b^2)(1+p^2+q^2)=0.$$

The corresponding contact transformation may now be set up. It is found that it is of the form

$$x_1-x=\frac{vp}{\sqrt{1+p^2+q^2}}\left[\frac{a^2-r^2}{a^2-v^2}\right]t,\quad y_1-y=\frac{vq}{\sqrt{1+p^2+q^2}}\left[\frac{b^2-r^2}{b^2-v^2}\right]t,$$

$$z_1-z=\frac{-v}{\sqrt{1+p^2-q^2}}\left[\frac{c^2-r^2}{c^2-v^2}\right]t,\quad p_1=p,\quad q_1=q,$$

where

$$r^2=\sum\frac{(x_1-x)^2}{t^2}.$$

Elimination of p and q from these equations gives the wave surface in the most general form, viz.:

$$\frac{(x_1-x)^2}{t^2(r^2-a^2)}+\frac{(y_1-y)^2}{t^2(r^2-b^2)}+\frac{(z_1-z)^2}{t^2(r^2-c^2)}=1.$$

For $x = y = z = 0$ and $t = 1$ (t being intrepreted as time) we have the wave surface as usually given.

The advantage of the above form consists in this that it puts directly in evidence Huygens' principle.

The wave motion in a biaxial crystal may be represented mathematically by a contact transformation whose characteristic function is

$$w = \sqrt{1 + p^2 + q^2}\, v,$$

v^2 being a root of the quadratic equation above.

10. In this paper Dr. Gillespie considers the definite integral

$$\int_a^b f(\alpha, x)dx,$$

where $f(\alpha, x)$ is an integrable * function of x for all values of α in a closed interval. The question to which an answer is sought is: what conditions must be imposed upon the $f(\alpha, x)$ so that

$$\left| \int_a^b f(\alpha, x)dx - \sum f(\alpha, x_i)(x_i - x_{i-1}) \right| < \epsilon \quad (i = 1, \cdots, n),$$

where $|x_i - x_{i-1}| < \eta$ for all values of α? η and ϵ have the usual significance, and the fixed law of subdivision is such that the limit of $x_i - x_{i-1}$ as n becomes infinite is zero. Some examples are given showing the necessity of imposing conditions on $f(\alpha, x)$; sufficient conditions are then derived. Supposing $f(\alpha, x)$ to be a continuous function of α for all values of x, the necessary and sufficient condition that

$$\int_a^b f(\alpha, x)dx$$

be a continuous function of α is obtained. A further application of this theory is found in certain problems in integral equations.

11. In this paper Professor Bowden gives a proof of the correctness of the Russian peasant method of multiplication, for the performance of which it is necessary to know only how to add, how to double a number, and how to divide by two,

* Riemann's definition of the definite integral.

obtaining the exact or lower approximate quotient. The method may be described as follows:

To multiply a by b write down $a \times b$. Under a write the exact or lower quotient obtained by dividing it by 2; under this quotient write the quotient obtained by dividing it by 2; and so on, until we obtain the quotient 1.

Under b write its double, under this double its double, and so on, until we have as many numbers in the second column as in the first.

Next add the numbers in the second column which correspond to odd numbers in the first. The result is the product of a and b.

The proof depends on the theorems given on pages 167, 168 in Part I of Chrystal's Text-Book of Algebra.

12. The theorem in question is proved by Dr. Lennes by setting the terms of the expansion of the determinant into one-to-one correspondence with the points of a line segment.

13. Professor Hancock's paper is in abstract as follows: If the three trinomials $x^2 + a_1x + 1$, $x^2 + a_2x + 1$, $x^2 + a_3x + 1$ are multiplied together and the coefficients of the resulting expression equated to those of $(x^7 - 1)/(x - 1)$, it is seen that $a_1 = -2\cos 2\pi/7$, $a_2 = -2\cos 4\pi/7$, $a_3 = -2\cos 6\pi/7$ are the roots of the equation $f_3(x) = x^3 - x^2 - 2x + 1 = 0$. In a similar manner are derived $f_4(x) = x^4 - x^3 - 3x^2 + 2x + 1 = 0$ (roots: $x = -2\cos 2\nu\pi/9$ $(\nu = 1, 2, 3, 4)$), $f_5(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$ (roots: $x = -2\cos 2\nu\pi/11$ $(\nu = 1, 2, 3, 4, 5)$), $\cdots$.

The following relations are at once evident:

$$f_k(2) = 1 \qquad (k = 1, 2, 3, \cdots, n),$$

$$f_k(0) = (-1)^{\frac{\tau^2-1}{2}} = \prod_{\nu=1}^{\nu=k} 2\cos\frac{2\nu m\pi}{\tau} \qquad \begin{pmatrix} m \text{ an odd integer} \\ \tau = 2k+1 \end{pmatrix}.$$

The latter formula is of importance in the theory of quadratic residues.

Write

$$\vartheta_\nu = -2\cos\frac{2\nu\pi}{2k+1},$$

where ν is any one of the integers $1, 2, \cdots, k$ and where *at*

first $2k+1$ is taken as a prime integer. The latter restriction is eventually removed.

Let $\Omega_k = R(\vartheta_\nu)$ be the realm of rationality determined by ϑ_ν, where, as seen above, ϑ_ν is a root of $f_k(x) = 0$. Several different ways are given for the determination of the discriminant of this realm. N denoting the norm of the quantity which follows it, it is found that

$$\Delta(1, \vartheta_\nu, \vartheta_\nu^2, \cdots, \vartheta_\nu^{k-1}) = (-1)^{\frac{k(k-1)}{2}} N\{f_k'(\vartheta_\nu)\} = (2k+1)^{k-1}.$$

It is also seen that this number is the basal invariant [Grundzahl $= \Delta(\Omega_k)$] of the realm Ω_k, the index of the above basis being unity. The quantities ϑ_ν are algebraic units in Ω_k; and, if $\mu = 2 + \vartheta_\nu$, then $1, \mu, \mu^2, \cdots, \mu^{k-1}$ form a basis of all algebraic integers in Ω_k.

Similar results are derived for the equations, whose roots are

$$\sigma_\nu = 2 \sin \frac{2\nu\pi}{2k+1} \quad (\nu = 1, 2, 3, \cdots, 2k;\ k = 1, 2, \cdots, n),$$

and the results which have hitherto been derived are compared with those which follow from the equations

$$(x^{2k+1} - 1)/(x - 1) = 0 \quad (k = 1, 2, \cdots, n).$$

14. A group of order p^m (p a prime) and class one, two, or three, can be simply isomorphic with irreducible homogeneous linear groups of only one degree. This is also true of those groups all of whose non-invariant commutators give invariant commutators besides identity. Moreover no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees. Professor Fite considers the question, suggested by the consideration of these facts, as to whether any group of order p^m can be simply isomorphic with irreducible groups of different degrees. The question is not answered, but a few facts having a bearing thereon are established.

F. N. COLE,
Secretary.

ON THE CLASSIFICATION OF CRYSTALS.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, December 29, 1910.)

THE determination of the various types of crystal symmetry is a problem which has attracted the attention of many physicists and mathematicians. The first solutions of the problem are those of Hessel,* of Bravais,† and of Gadolin,‡ while the latest and most elegant is that of Lorentz.§ In what follows I should like to show that by combining theorems due to Curie and to Lorentz it is very easy to enumerate and to remember the different types of symmetry.

A few preliminaries are necessary. In the first place, if the line joining two points P and P' is bisected by a point O, we shall say that P' is the inverse of P with respect to O. In the next place we shall use the phrase inversion with respect to a point to denote the operation which consists in replacing every point of a figure by its inverse with respect to the given point.

Let us consider any system of points. If the system is such that rotation through a suitable angle about a line OA transforms the system of points into itself, we shall say that OA is an axis of direct symmetry. It is obvious that if rotation through an angle α transforms the system into itself, rotation through any multiple of α will also transform the system into itself. It follows that the number of different rotations which transform the system into itself will be infinite unless α is commensurable with 2π. Accordingly, if we limit ourselves to systems which admit of a finite number only of transformations, α must be equal to $2\pi m/q$, where m and q are integers prime to each other.

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*J. F. C. Hessel, Krystallometrie oder Krystallonomie und Krystallographie; Gehler's Physikalisches Wörterbuch, vol. 5, 1830, p. 1062; Ostwald's Klassiker der exakten Wissenschaften, no. 88.

†A. Bravais, "Mémoire sur les polyèdres de forme symétrique," *Journal de mathématiques pures et appliquées*, vol. 14 (1849), p. 141; Ostwald's Klassiker der exakten Wissenschaften, no. 17.

‡A. Gadolin, "Mémoire sur la déduction d'un seul principe de tous les syatèmes crystallographiques," *Acta Societatis Scientiarum Fennicae*, Helsingfors, vol. 9 (1871), p. 1; Ostwald's Klassiker der exakten Wissenschaften, no. 75.

§H. A. Lorentz, "Über die Symmetrie der Kristalle", Abhandlungen über theoretische Physik, vol. 1 (1907), p. 299.

Morever, it can be shown without difficulty that the different rotations about OA correspond to the first $q-1$ multiples of the angle $2\pi/q$. The axis OA is said to be of order q, and the rotations about it may be performed in either direction. It is clear that if OA is an axis of direct symmetry of order q, the same is true of OA', the inverse of OA with respect to O.

The system of points may be such that it is possible to draw through a point O two sets of rectangular coordinate axes, one right-handed and the other left-handed, in such a way that to every point of the system whose coordinates with respect to one set of axes are the numbers x, y, z, there corresponds a point of the system whose coordinates with respect to the other set of axes are the same numbers x, y, z. When this is the case, it is easy to see that it is possible to transform the system into itself by means of a rotation followed by an inversion. Indeed, it is clear that a rotation can be found which will bring one set of coordinate axes into coincidence with the inverses of the other set; this rotation, followed by an inversion with respect to O, will obviously transform every point of the system into a point of the system. If the rotation is unnecessary, so that a mere inversion with respect to O transforms the system into itself, we shall say that O is a center of symmetry. In the general case, where a rotation is necessary, we shall call the axis of rotation an axis of inverse symmetry.

If, as before, we limit ourselves to systems which admit of a finite number only of transformations, it follows that the angle of rotation associated with an axis of inverse symmetry must be equal to $2\pi m/q$, where m and q are integers prime to each other. If q is odd, it can be shown that the axis of inverse symmetry is equivalent to an axis of direct symmetry of order q and a center of symmetry; if, on the contrary, q is even, the various operations connected with the axis of inverse symmetry are equivalent to the repetition of a rotation through an angle $2\pi/q$ accompanied by an inversion.

To establish this, we observe that the rotation about the axis of inverse symmetry may precede or follow the inversion; the result in each case is the same. Moreover, two successive inversions always transform a system into itself. Accordingly, q rotations through an angle $2\pi m/q$, accompanied by q inversions, are equivalent to rotation through an angle $2\pi m$ followed by q inversions; if q is odd, the operation is thus equivalent to a single inversion and the system possesses a center of sym-

metry. From this it follows immediately that a mere rotation through the angle $2\pi m/q$ transforms the system into itself. Accordingly, an axis of inverse symmetry of odd order is equivalent to an axis of direct symmetry of the same order and a center of symmetry.

Again, the repetition k times of the operation of rotation through an angle $2\pi m/q$ and inversion is equivalent to a single rotation of angle $2\pi mk/q - 2\pi l$, where l is an integer, followed by k inversions. Since

$$\frac{2\pi mk}{q} - 2\pi l = \frac{2\pi}{q}(mk - ql),$$

and since we can always choose k and l so that

$$mk - ql = \pm 1,$$

it follows that one of the operations connected with the axis of inverse symmetry consists in a rotation of angle $2\pi/q$ followed by k inversions. From the last equation it follows that k must be prime to q; accordingly, when q is even, k must be odd. Thus, when q is even, all the operations connected with the axis of inverse symmetry are repetitions of a rotation through an angle $2\pi/q$ accompanied by an inversion. There are $q-1$ different operations and no one of them is an inversion.

It is now easy to see that the crystallographic problem of the determination of the various possible types of crystal symmetry is equivalent to the problem of determining the various sets of axes of direct and of inverse symmetry passing through a common point, which a system can possess; the point of intersection of the axes may or may not be a center of symmetry.

For this purpose it is sufficient to recall that crystals possess two kinds of axes of symmetry. An axis of symmetry of the first kind is characterized by the fact that the physical properties of the crystal with respect to any fixed directions in space remain unchanged when the crystal is turned through a suitable angle about the axis. Accordingly, if we consider a portion of the crystal bounded by a spherical surface and if we draw the axis of symmetry through the center of this sphere, we may say that a rotation about the axis transforms the crystal into itself.

An axis of symmetry of the second kind is characterized by the fact that if the crystal is turned through a suitable angle about such an axis, its physical properties in its new position

and with respect to any fixed directions in space are identical with the properties of the crystal in its first position and with respect to the fixed directions which are the inverse of those just mentioned. Accordingly, if, as before, we consider a portion of the crystal bounded by a spherical surface, and if, through the center, we draw an axis of symmetry of the second kind, we may say that a rotation about this axis, followed by an inversion with respect to the center of the sphere, transforms the crystal into itself.

We shall now proceed to the enumeration of the different sets of axes of direct and of inverse symmetry, passing through a common point, that a system can possess. From what precedes, it is obvious that this problem is identical with the problem of finding all the finite groups of operations that can be generated by rotations about a point and inversion with respect to it.

Let us begin by finding the groups that consist of rotations only; it will then be easy to find the groups that contain inversions also. In the first place, it is clear that, if OA be an axis of direct symmetry of order q, the different positions which OA takes when the system is subjected to all the rotations of the group are also axes of direct symmetry of order q. Every axis thus belongs to a definite set of equivalent axes and the various axes of the system can be grouped into one or more sets of equivalent axes. In enumerating the axes of direct symmetry we must distinguish between OA and its inverse OA'; OA and OA' will belong to the same or to different sets of equivalent axes according as it is or is not possible to bring OA into coincidence with OA'.

We can now establish the following two theorems which are due to Curie* and which suffice to determine the possible groups of rotations.

THEOREM I. *If n be the number of different rotations, including the identical rotation, which transform a system of points into itself, and if p be the number of different positions which an axis of order q assumes when the system is subjected to these n rotations, then*

$$pq = n. \tag{1}$$

*P. Curie, "Sur les questions d'ordre," *Bulletin de la Société minéralogique de France*, vol. 8 (1884), p. 89; Oeuvres, p. 70. "Sur les répétitions et la symétrie," *Comptes rendus de l'Académie des Sciences*, vol. 100 (1885), p. 393; Oeuvres, p. 114.

THEOREM II. *If a system possesses a set of p equivalent axes of order q, a set of p' equivalent axes of order q', a set of p'' equivalent axes of order q'', $\cdots$, and if k be the number of these sets, then*

$$(2) \quad p(q-1)+p'(q'-1)+p''(q''-1)+\cdots=2(n-1),$$

or

$$(3) \quad p+p'+p''+\cdots=(k-2)n+2,$$

or finally

$$(4) \quad \frac{1}{q}+\frac{1}{q'}+\frac{1}{q''}+\cdots=k-2+\frac{2}{n}.$$

To establish equation (1) it is sufficient to observe that for each of the p positions which the axis of rotation under consideration can assume, the system can assume any one of q positions obtained by rotation about this axis; the system can thus be transformed into itself by any one of pq different operations. Moreover, it is obvious that the pq different positions thus obtained are the only ones which the system can take; thus pq is equal to n.

To establish equation (2) we observe that the system can be transformed into itself by any one of $q-1$ rotations about each of the p axes of the first set, by any one of $q'-1$ rotations about each of the p' axes of the second set, by any one of $q''-1$ rotations about each of the p'' axes of the third set, $\cdots$, and that in this enumeration every rotation, except the identical rotation, has been counted twice; OA and its inverse OA' may belong to the same or to different sets of axes, in either case the rotations about AA' have been counted twice. Equation (3) follows without difficulty from equations (1) and (2,) and in like manner equation (4) follows from equations (1) and (3).

The solutions of equations (1) and (4) will give us all the sets of axes of direct symmetry that can coexist. There is no difficulty in finding these solutions; it is found that when $k=1$ or $k>3$ there are no solutions, and the results for $k=2$ and $k=3$ are given in the following table:

k	p	p'	p''	q	q'	q''	n
2	1	1		n	n		n
3	m	m	2	2	2	m	$2m$
	6	4	4	2	3	3	12
	12	8	6	2	3	4	24
	30	20	12	2	3	5	60

We shall now show that corresponding to each solution in the above table there exists one and but one arrangement of the axes of symmetry in space. When $k = 2$, there are two sets of equivalent axes each consisting of a single axis of order n; it is obvious that these two axes must be inverses of each other. The geometric configuration corresponding to this solution thus exists and it is unique. The corresponding groups of rotations are called the cyclic groups.

When $k = 3$, let us draw a unit sphere about O as a center and let us mark on the sphere the points where the three sets of axes of symmetry pierce it. It is well known that if we construct any spherical triangle ABC, a rotation about OA through an angle $2A$, followed by a rotation about OB through an angle $2B$, is equivalent to a rotation about OC through an angle $2C$. Accordingly, if OA and OB are axes of symmetry of orders q and q' and if the angles A and B are respectively equal to π/q and π/q', the line OC will also be an axis of symmetry of the system and the half-angle of rotation associated with it will be equal to C or to some aliquot part of C. We shall establish the following theorem:

THEOREM III. *If OA and OB are axes of symmetry of orders q and q' belonging to different sets of equivalent axes and if the angle between OA and OB is not greater than the angle between any two axes of symmetry belonging to different sets, then, if we construct the spherical triangle ABC with angles A and B respectively equal to π/q and π/q', the line OC will be an axis of symmetry of order q'' belonging to the third set of equivalent axes and the angle C will be equal to π/q''.*

In the first place, the half-angle of rotation associated with the axis OC is equal to the angle C and not to an aliquot part

of C. For if an aliquot part of C were the half-angle of rotation associated with the axis OC, we could draw through C an arc making this angle with AC and thus form a triangle ACD; the line OD, lying between OA and OB, would thus, contrary to hypothesis, be an axis of symmetry.

In the next place, it is obvious that if the angle C differs from the angles A and B, the axis of symmetry OC belongs to the third set of equivalent axes. It remains to prove that this is the case even when the angle C is equal to one of the angles A or B. Accordingly, let us suppose that in the triangle ABC the angles B and C are equal and that OB and OC belong to the same set of equivalent axes. It is easy to see that no axis belonging to the third set of axes can pierce the sphere within the triangle ABC or on either of the sides AB and AC, for any such axis would make with OA a smaller angle than does OB. This becomes evident if we remember that no one of the angles of the triangle is greater than $\frac{1}{2}\pi$ and that consequently the perpendiculars from the vertices to the opposite sides do not fall without the triangle. Further, no axis of the third set can meet the side BC. The reason just given applies here also except when AB and AC are quadrants; but in that case the arc BC cannot be greater than a quadrant since the angle A is not greater than $\frac{1}{2}\pi$, and consequently any axis meeting the side BC would form with OB an angle less than the angle between OB and OA. Thus, no axis belonging to the third set can pierce the sphere within the triangle ABC or on its perimeter.

Let us now construct a triangle BCA_1 equal to BCA and having the side BC in common. The vertex A_1 corresponds to an axis belonging to the same set as OA, for OA_1 is the position which OA takes when it is revolved about OB through an angle $2B$. In like manner, we can construct triangles ACB_1 and ABC_1 equal to ABC; the vertices B_1 and C_1 will correspond to axes belonging to the same set as OB and OC. By repeating this construction we obtain a network of triangles the vertices of which correspond to axes belonging to the same sets as OA, OB, OC. This network must cover the sphere completely and there can be no overlapping. Indeed, it is obvious that the construction can be continued as long as any portion of the surface is uncovered. On the other hand there can be no overlapping, for, if there were, there would lie within a triangle such as ABC a point D corresponding to an axis belonging to

one of the first two sets. If we remember that the angles A, B, C are not greater than $\frac{1}{2}\pi$ and that consequently the perpendiculars from the vertices to the opposite sides do not fall without the triangle, it is clear that DA and DB are each less than AB. We should then have, contrary to the hypothesis, an axis nearer to OA or to OB than these two are to each other.

The sphere has thus been covered by a network of triangles equal to ABC, the vertices of which correspond to axes of the first and second sets. Moreover, it has been shown that no axis of the third set can pierce the triangle ABC. In like manner it can be shown that no axis of the third set can pierce a triangle belonging to the network of equal triangles. There is thus no place for the axes belonging to the third set. We must therefore abandon the hypothesis that OC is an axis belonging to the second set and our Theorem III is established.

Let us now consider a triangle ABC of the kind described in Theorem III. By means of the construction described above we can obtain a set of triangles equal to or symmetrical with ABC and the vertices of these triangles will correspond to axes belonging to the three sets of axes. Moreover, it can be shown as above that this network of triangles will cover the sphere completely and that there will be no overlapping. We are thus led to the problem of determining the values of q, q', q'' such that we can cover the sphere with a set of triangles whose angles are π/q, π/q', π/q''. Since the area of one of these triangles is

$$\frac{\pi}{q}+\frac{\pi}{q'}+\frac{\pi}{q''}-\pi,$$

we must have, if N denote the number of triangles,

$$N\left(\frac{\pi}{q}+\frac{\pi}{q'}+\frac{\pi}{q''}-\pi\right)=4\pi,$$

or

$$\frac{1}{q}+\frac{1}{q'}+\frac{1}{q''}=1+\frac{4}{N}. \tag{5}$$

As this equation differs from equation 4 for $k=3$ only in the substitution of N for $2n$, its solutions can be at once written down. They are given by the following table

q	q'	q''	N
2	2	m	$4m$
2	3	3	24
2	3	4	48
2	3	5	120

It is obvious that if there exists a network of triangles corresponding to any one of these solutions it is unique; for the triangle ABC is determined when its angles are given and the other triangles are determined without ambiguity when ABC is given. Now it is well known that there exists a set of triangles corresponding to each of the above solutions. The $4m$ triangles corresponding to the first solution are obtained by dividing the circumference of a great circle into $2m$ equal parts and connecting the points of division with the poles of the circle. The radii to the poles of the great circle form a set of 2 equivalent axes of order m, while the $2m$ radii to the points of division of the great circle fall into two sets of m equivalent axes of order 2. The groups of rotations corresponding to this case are called the dihedral groups.

The configuration corresponding to the second solution in the table is found by inscribing in the sphere a pair of polar tetrahedrons. Each vertex of the first tetrahedron lies within a triangle formed by vertices of the other; by connecting each vertex of the first tetrahedron with the vertices and the mid-points of the sides of the surrounding triangle we find 24 triangles whose angles are $\pi/2$, $\pi/3$, $\pi/3$. The vertices of the two tetrahedrons correspond to the two sets of 4 equivalent axes of order 3, while the remaining vertices of the triangles correspond to the 6 axes of order 2. The corresponding group of rotations is called the tetrahedral group.

The configuration corresponding to the third solution in the table is found by inscribing in the sphere a cube and its polar octahedron. Each vertex of the octahedron lies within a quadrilateral formed by vertices of the cube; by connecting each vertex of the octahedron with the vertices and the mid-points of the sides of the surrounding quadrilateral we find 48 triangles whose angles are $\pi/2$, $\pi/3$, $\pi/4$. The vertices of the octahe-

dron correspond to a set of 6 equivalent axes of order 4, the vertices of the cube correspond to a set of 8 equivalent axes of order 3, while the remaining vertices of the triangles correspond to the 12 axes of order 2. The corresponding group of rotations is called the octahedral group.

Finally, the configuration corresponding to the last solution in the table is found by inscribing in the sphere a regular dodecahedron and its polar icosahedron. Each vertex of the icosahedron lies within a pentagon formed by vertices of the dodecahedron; by connecting each vertex of the icosahedron with the vertices and the mid-points of the surrounding pentagon we find 120 triangles whose angles are $\pi/2, \pi/3, \pi/5$. The vertices of the icosahedron correspond to a set of 12 equivalent axes of order 5, the vertices of the dodecahedron correspond to a set of 20 equivalent axes of order 3, while the remaining vertices of the triangles correspond to the 30 axes of order 2. The corresponding group of rotations is called the icosahedral group.

Thus the only groups of rotations are the identical rotation and the cyclic, the dihedral, the tetrahedral, the octahedral, and the icosahedral groups.

As Lorentz * has shown, we can now obtain without difficulty the groups of operations that contain inversions as well as rotations. In the first place, we may associate an inversion with each of the groups of rotations; the system will then possess in addition to the axes of symmetry a center of symmetry.

In the next place, since an axis of inverse symmetry of odd order is equivalent to an axis of direct symmetry of the same order and a center of symmetry, we may now limit ourselves to the consideration of axes of inverse symmetry of even order. Furthermore, since the inversions and rotations about the axes of inverse symmetry must, like the rotations about the axes of direct symmetry, transform the group of axes of symmetry into itself, it follows that the only admissible configurations for the axes of symmetry are the configurations which we found in studying the groups of rotations. The groups of operations containing operations corresponding to axes of inverse symmetry will therefore be found by replacing one or more sets of axes of direct symmetry by axes of inverse symmetry of the same orders. Finally, if we consider a triangle of the kind

* L. c.

described in Theorem III, it is easy to see that if one of the three axes OA, OB, OC be replaced by an axis of inverse symmetry, one other must also be replaced by such an axis while the third will remain an axis of direct symmetry.

With these remarks in mind it is easy to enumerate the groups of operations which contain operations corresponding to axes of inverse symmetry. In the first place, we obtain from every cyclic group of even order a new group by replacing the axis of direct symmetry by an axis of inverse symmetry of the same order. In the next place, we obtain from every dihedral group a new group by replacing the two sets of axes of order 2 by axes of inverse symmetry of order 2. Moreover, we obtain a new group from every dihedral group of order $2m$, when m is an even number equal to or greater than 4, by replacing the pair of axes of order m and one set of axes of order 2 by an axis of inverse symmetry of order m and axes of inverse symmetry of order 2. Finally, we obtain a new group from the octahedral group by replacing the axes of orders 2 and 4 by axes of inverse symmetry of the same orders. The tetrahedral and icosahedral groups furnish no new groups.

Our results are summarized in the following table, in which the letters C, D, T, O, I are used to recall the cyclic, the dihedral, the tetrahedral, the octahedral, and the icosahedral groups and the subscripts are used to indicate the orders of the corresponding axes of symmetry. A bar over a subscript indicates that the corresponding axes of direct symmetry have been replaced by axes of inverse symmetry.

TYPES OF SYMMETRY CHARACTERIZED BY THE EXISTENCE OF

Axes of Direct Symmetry.	Center of Symmetry and Axes of Direct Symmetry.	Axes of Direct and of Inverse Symmetry.
No axis	No axis	
C_n	C_n	$C_{\overline{2m}}$
$D_{2,2,m}$	$D_{2,2,m}$	$D_{\overline{2},\overline{2},m}$ $D_{2,2,\overline{2(\kappa+2)}}$ $\kappa \geqq 0$
$T_{2,3,3}$	$T_{2,3,3}$	
$O_{2,3,4}$	$O_{2,3,4}$	$O_{\overline{2},3,\overline{4}}$
$I_{2,3,5}$	$I_{2,3,5}$	

Not all of the types of symmetry enumerated in this table are available as types of crystal symmetry, for the law of rational indices limits the acceptable axes of symmetry to those of the orders 2, 3, 4, 6. With this limitation the table furnishes the 32 types of crystal symmetry, 11 from each of the first two columns and 10 from the third.

NEW YORK.
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HORNER'S METHOD OF APPROXIMATION ANTICIPATED BY RUFFINI.

BY PROFESSOR FLORIAN CAJORI.

(Read before the Southwestern Section of the American Mathematical Society, November 26, 1910.)

BEFORE the nineteenth century no simple elementary practical process was known of computing the coefficients of an equation whose roots are less by a given constant than the roots of a given affected numerical equation $f(x) = 0$. Such a process was invented in the early part of the last century and is contained in the so-called "Horner's method of approximation."

The history of Horner's method, as developed in England, has been traced in detail by Augustus De Morgan.* He quotes extensively from original sources and shows that, of English rivals (among whom were Theophilus Holdred, Peter Nicholson, and Henry Atkinson), none except perhaps Nicholson advanced methods of approximation that equalled Horner's, and none was entitled to priority over Horner. It is well-known that the popularization of Horner's process of approximation in England was due to De Morgan and J. R. Young. Except for the efforts of these men, Horner's paper of 1819 in the *Philosophical Transactions* might have been lost sight of and forgotten. De Morgan was an enthusiast on Horner's method. He taught it with great zeal; he made sport of Cambridge tutors who were not familiar with it;† the preparation of his historical tract, alluded to above, was evidently a labor

* *Companion to the* [*British*] *Almanac for 1839*, Art. "Notices of the progress of the problem of evolution," pp. 34–52.

† A. De Morgan, A Budget of Paradoxes, London, 1872, pp. 292, 375.

of love. It is still the standard reference to early English research on the subject. That Horner had a rival claimant outside of England, De Morgan never suspected.*

It is the purpose of this paper to show that fifteen years before the publication of Horner's first paper† an almost identical method was published in Italy by Paolo Ruffini, whose name is usually associated with the proof of the impossibility of solving algebraically the quintic equation. Ruffini had no J. R. Young and no De Morgan to urge upon the mathematical public the importance of his method of approximation. The method was overlooked, and now, over a century later, begins the weary process of the "restitution of decayed intelligence."

In 1802 the Italian Scientific Society (of Forty) offered a gold medal for the best method of determining the roots of a numerical equation of any degree. Five dissertations were submitted in competition and in 1804 the medal was awarded to Paolo Ruffini. The dissertation was published by the Society in an unwieldy, large quarto monograph of 175 pages, bearing the title:

"Sopra la determinazione delle radici nelle equazioni numeriche di qualunque grado. Memoria del Dottor Paolo Ruffini, pubblico professore di matematica sublime in Modena, uno dei quaranta della società italiana delle scienze ec. Coronata dalla società medesima. In Modena, MDCCCIV. Presso la società tipografica. Con Approvazione."

Ruffini wrote on the subject of approximation again in 1807‡ and 1813.§ In these later articles he gave a simpler exposition of the method and explained its application to the extraction of roots of numbers.

While Horner had no knowledge of Ruffini's publications on

* Budget of Paradoxes, p. 375.

† Horner prepared three important papers on his method of approximation: (1) "A new method of solving numerical equations of all orders, by continuous approximation," *Philosophical Transactions* (London), 1819, Part I, pp. 308–335; (2) An article in *Leybourn's Mathematical Repository*, Vol. 5, Part 2, No. 19, London, 1830; Extracts from this are given in J. R. Young's Theory and Solution of Algebraical Equations, London, 1843, pp. 474, 475; (3) "On algebraic transformation," *The Mathematician*, Vol. I, London, 1845, p. 109.

‡ Algebra elementare del Dottor Paolo Ruffini, Modena, MDCCCVII, Capo IV, V, and in an Appendice which is not in my copy of this Algebra and seems to have been bound separately and published in 1808.

§ "Di un nuovo metodo generale di estrarre le radici numeriche." Memoria del Signor Cav. Paolo Ruffini. *Memorie di Matematica e di Fisica della Società Italiana delle Scienze.* Tomo XVI, Parte I. Verona, MDCCCXIII, pp. 373–429; Tomo XVII, parte contenente del memorie di matematica, pp. 1–15.

the approximation to the roots of equations, Horner's papers of 1819 and 1845 contain nevertheless some interesting parallelisms to Ruffini's publications of 1804, 1807, and 1813.

In their first publications both authors used the differential calculus in expounding their methods; Ruffini used ordinary derivatives, Horner used Arbogast's derivatives.

In later publications both authors gave simplified explanations of their methods, dispensing with the calculus and using ordinary algebra. Ruffini's simplified exposition appeared in 1807 and 1813; Horner's in 1845 as a posthumous article.*

Both authors pointed out that the methods could be used with advantage in the root-extraction of numbers. Ruffini laid greater stress on this point than did Horner; in fact, Ruffini's publications of 1807 and 1813 consider the process from this point of view alone. Neither author extended the method of approximation so as to render it applicable to the determination of complex roots.

A contrast between the two authors lies in the fact that Horner in 1819 erroneously believed that his method was at once applicable to irrational and transcendental equations. He says: "From the unrestricted nature of the notation employed, it is evident that no class of equations, whether finite, irrational, or transcendental, is excluded from our design."† He gives no other argument nor any illustrative solutions of irrational or transcendental equations, in support of his claim. Ruffini, on the other hand, confined his discussion to rational integral algebraic equations.

A second contrast is this, that Horner in his paper of 1830 carefully explains a special procedure in his method, by which two nearly equal real roots can be separated with certainty. Ruffini never considered this refinement of the method.

To explain Ruffini's method of approximation, we quote from pages 22–25 of his memoir of 1804. The equation (A), referred

* This article of 1845 had been read before the Royal Society, June 19, 1823, but (as T. S. Davies explains in an introductory note to the printed paper) was refused publication in the *Philosophical Transactions*, even after it had been considerably curtailed. Horner often stated to Davies that much demur was made to the insertion of his first paper in the *Philosophical Transactions*. "The *elementary character* of the subject," says Davies, "was the professed objection: his recondite mode of treating it, was the professed passport for its admission." Horner's third article was a simplification of the principles of his process. In this article he says of Budan, I am "satisfied with finding that he had not anticipated my discovery."

† *Philosophical Transactions*, 1819, Part I, pp. 317, 318.

to in the quotation, is as follows:

$$(A) \quad Ax^m + Bx^{m-1} + Cx^{m-2} + \text{ec.} + Sx^2 + Tx + V = 0.$$

"Supposto $x = p + y$, determinare un metodo spedito, onde trasformare l' Equazione (A) in un' altra con l' incognita y.

"Dalle proprietà delle trasformate sappiamo, che, essendo Z il primo membro della (A) (n.° 1), la trasformata richiesta si ottiene ponendo p in luogo di x nella

$$Z + \frac{dZ}{dx}y + \frac{ddZ}{2dx^2}y^2 + \frac{d^3Z}{2\cdot 3dx^3}y^3 + \text{ec.} + Ay^m = 0.$$

Ora pel (n.° 10) nella ipotesi di $x = p$ abbiamo $Z = P^{(m+1)}$, e quindi

$$\frac{dZ}{dx} = \frac{dP^{(m+1)}}{dp} = Q^{(m)} \text{ (n.° 8)}, \quad \frac{ddZ}{2dx^2} = \frac{dQ^{(m)}}{2dp} = R^{(m-1)},$$

$$\frac{d^3Z}{2\cdot 3dx^3} = \frac{dR^{(m-1)}}{3dp} = S^{(m-2)}, \text{ ec.}$$

Dunque queste quantità $P^{(m+1)}$, $Q^{(m)}$, $R^{(m-1)}$, $S^{(m-2)}$, ec. altro non sono che i coefficienti della Equazione in y, ed essa sarà perciò la seguente

$$(F) \quad P^{(m+1)} + Q^{(m)}y + R^{(m-1)}y^2 + S^{(m-2)}y^3 + \text{ec.} + Ay^m = 0.$$

"Ciò essendo, per isciogliere il nostro Problema, altro non dovremo, che determinare speditamente le quantità $P^{(m+1)}$, $Q^{(m)}$, $R^{(m-1)}$, $S^{(m-2)}$, ec.

"A tal fine disposti in (IV) i coefficienti della (A) nel modo istesso del (I) (n.° 9), e determinate nella maniera medesima tutte le quantità A, P'', P''', ec. $P^{(m+1)}$, colloco in una terza riga sotto della P'' il coefficiente A, moltiplico questo per p, sommo il prodotto $Ap = Q'p$(2°. n.° 8) col sovrapposto P'', e il risultato $Q'p + P''$, essendo pel cit.° (2°. n°. 8) $= Q''$, si ponga nella terza linea sotto dell termine P'''. Si moltiplici questo Q'' per p, si unisca con P''', e pongasi sotto del P^{IV} il risultato $Q''p + P''' = Q'''$(2°. n°. 8). Proseguendo così ad operare, otterremo sotto del $P^{(m+1)}$ il risultato $Q^{(m)}$....

$$
\begin{array}{lccccccc}
 & A, & B, & C, & D, & E, & F, & \text{ec. } V, \\
 & & A, & P'', & P''', & P^{IV}, & P^{V}, & \text{ec. } P^{(m)}, \quad P^{(m+1)}, \\
 & & & A, & Q'', & Q''', & Q^{IV}, & \text{ec. } Q^{(m-1)}, \quad Q^{(m)}, \\
\text{(IV)} & & & & A, & R'', & R''', & \text{ec. } R^{(m-2)}, \quad R^{(m-1)}, \\
 & & & & & A, & S'', & \text{ec. } S^{(m-3)}, \quad S^{(m-2)}, \\
 & & & & & & A, & \text{ec. } T^{(m-4)}, \quad T^{(m-3)}, \\
 & & & & & & & \text{ec.}
\end{array}
$$

"Nell' ultima $(m+1)$ *esima* colonna verticale contenendosi i valori dei coefficienti della (F), li sostituisco nella (F) medesima, e per tal guisa otterremo la trasformata richiesta con operazioni semplici, e facili ad eseguirsi, come ognuno può agevolmente vedere da se, e come apparisce dall' esempio seguente.

"Data la $4x^5 - 6x^4 + 3x^3 - 5x^2 - 4x + 8 = 0$, vogliasi trasformare in un' altra, nella quale sia $y = x - 6$. Opero nel modo ora esposto,

4,	-6,	3,	-5,	-4,	8,	
	4,	18,	111,	661,	3962,	23780
		4,	42,	363,	2839,	20996
			4,	66,	759,	7393
				4,	90,	1299
					4,	114
						4

e otterremo la trasformata

$$4y^5 + 114y^4 + 1299y^3 + 7393y^2 + 20996y + 23780 = 0."$$

It appears from this that the computation demanded by Ruffini is identical with that in the familiar process of Horner. But the arrangement of the figures is different. The coefficients of the transformed equation appear, with Ruffini in the column on the extreme right, with Horner along a diagonal line slanting upward from left to right. It is not evident that, of the two arrangements, one possesses any marked superiority over the other. On pages 77–79 Ruffini computes the real root of Newton's cubic, $x^3 - 2x - 5 = 0$, to eight decimal places.

Ruffini in 1804 elaborated "Horner's method" of approximation to the roots of numerical equations with a clearness and thoroughness not surpassed in Horner's own exposition of 1819. In view of this fact, does not historical justice demand that the name of Ruffini be associated with that of Horner in

the designation of the method? Why not call it the "Ruffini-Horner method"?

It should be stated here that Budan, in his famous pamphlet of 1807,* explained a process of transformation of an equation into another whose roots are diminished by h, which bears some resemblance to that of Ruffini and Horner. When the roots are to be diminished by unity, the computation is identical. But when a root is to be diminished by, say 8, there is a difference: Budan does this by eight steps, the roots being diminished by unity in each step. Regarding Budan, Ruffini once wrote to Delambre, the secretary of the French Institute, as follows:†

"Troppo giuste sono le lodi che Ella dà al metodo del Sig. Budan di sciogliere le Equ. numeriche; avrei però desiderato che avesse Ella avuta occasione di vedere la Memoria che sopra lo stesso argomento presentai già alla nostra Società Italiana, che da essa riportò graziosamente il premio e che fu poi stampata nel 1804 fuori degli Atti. Coincido col sig. Budan nella maniera di fare le trasformazioni successive e nel servirmi dei decimali."

THE NEW HAVEN COLLOQUIUM LECTURES.

The New Haven Colloquium. By ELIAKIM HASTINGS MOORE, ERNEST JULIUS WILCZYNSKI, MAX MASON. Yale University Press, 1910. x + 222 pp.

THE fifth Colloquium of the American Mathematical Society was held at New Haven, September 5–8, 1906, under the auspices of Yale University. All the lectures related to fields in which recent progress has been considerable, and were given by men who have made important contributions; on this account the volume which contains the lectures should be of substantial interest, particularly to the American mathematician. Professor Moore gives a first systematic account of what he has termed "a form of General Analysis." Professor Wilczynski outlines the point of view and some of the principal results in

*F. D. Budan, Nouvelle méthode pour la résolution des équations numériques, Paris, 1807, pp. 14, 15, 29, 39.

† *Memorie della Società Italiana delle Scienze* (detta dei XL), Serie 3ª, Tomo XIV, 1906, p. 296.

projective differential geometry. Professor Mason treats a variety of boundary value problems.

I. *Introduction to a Form of General Analysis.*

It is obvious to those who have been following recent mathematical progress that, since the researches of Hill, Volterra, and Fredholm in the direction of extended linear systems of equations, mathematics has been in the way of a great development. That attitude of mind which conceives of the function as a generalized point, of the method of successive approximation as a Taylor's expansion in a function variable, of the calculus of variations as a limiting form of the ordinary algebraic problem of maxima and minima is now crystallizing into a new branch of mathematics under the leadership of Pincherle, Hadamard, Hilbert, Moore, and others. For this field Professor Moore proposes the term "General Analysis," defined (page 9) as "the theory of systems of classes of functions, functional operations, etc., involving at least one general variable on a general range." He has fixed attention on the most abstract aspect of this field by considering functions of an absolutely general variable. The nearest approach to a similar investigation is due to Fréchet (Paris thesis, 1906), who restricts himself to variables for which the notion of a limiting value is valid.

In the General Analysis we consider a class $\mathfrak{M}$ of real single valued functions ϕ_p of the variable p; important illustrative cases are:

I. $p = 1, 2, \cdots, n$; $\phi_p \equiv (\phi_1, \phi_2, \cdots, \phi_n)$, where $\phi_1, \cdots, \phi_n$ are arbitrary real quantities.

II. $p = 1, 2, \cdots, n, \cdots$; $\phi_p \equiv (\phi_1, \phi_2, \cdots, \phi_n, \cdots)$, where $\phi_1, \phi_2, \cdots$ are restricted in that $\lim_{p=\infty} \phi_p = 0$.

III. The same as II, except that the convergence restriction is now that $\sum_{p=1}^{\infty} \phi_p^2$ converges.

IV. $0 \leqq p \leqq 1$; ϕ_p is any real continuous function of p.

The class $\mathfrak{M}$ is linear (L) if the sum of every two functions of $\mathfrak{M}$ or the product of one such function by any real constant is in $\mathfrak{M}$.

The class $\mathfrak{M}$ is closed (C) if the limit ϕ_p of a convergent sequence $\alpha_p, \beta_p, \cdots$ of functions of $\mathfrak{M}$ is itself a function of $\mathfrak{M}$ whenever there exists a function λ_p of $\mathfrak{M}$ such that the difference between the successive members of the sequence and

the limit ϕ_p becomes and remains uniformly not greater in absolute value than $\epsilon\lambda_p$, where ϵ is an arbitrary small positive quantity. This mode of convergence is termed relatively uniform convergence as to the scale function λ_p, and becomes uniform convergence if the scale function is a constant.

The class $\mathfrak{M}$ possesses the dominance property D if for every finite or infinite sequence $\alpha_p, \beta_p, \cdots$ in $\mathfrak{M}$ there exists a dominating function λ_p of $\mathfrak{M}$ such that each particular element of the sequence does not exceed in absolute value a suitable constant multiple of λ_p for any p.*

Finally if the absolute value of any function in the class $\mathfrak{M}$ is a function of $\mathfrak{M}$, that class is said to be absolute (A).

It is clear that in the cases I, II, III, IV the class of functions has the properties L, A and it is also easily seen that C, D will hold (cf. Theorem, page 42).

The property C in case IV is equivalent to the property that the limit of a uniformly converging sequence of continuous functions is itself a continuous function. In case II, C may be seen to hold as follows: Suppose we have a sequence of sequences each converging to zero, like terms of successive sequences converging to the like terms of a limit sequence, and furthermore in such a way that the term-by-term difference from the limit becomes and remains less than ϵ times the corresponding term of some sequence converging to zero. The limit sequence will then have the property of converging to zero since its terms are less in absolute value than the sum of the absolute values of the corresponding terms of two sequences converging to zero. In case III, C also holds and to demonstrate this fact we need merely to modify the above argument by replacing the condition 'converging to zero' by the condition 'with sum of squares convergent,' and to note that the sum of squares of the elements of the limit sequence is less than twice the sum of squares of the elements of two sequences in $\mathfrak{M}$ by virtue of the inequality

$$(a + b)^2 \leqq 2(a^2 + b^2).$$

The property D holds in case IV when we may take $\lambda_p = 1$ as the dominating function for any sequence; in case II if $M_1, M_2, \cdots$ denote positive quantities greater than any term of

* The word 'dominating' is not used in the Lectures with the same significance.

the sequence of $\alpha_p, \beta_p, \cdots$ respectively in absolute value, we may take

$$\lambda_p = \frac{|\alpha_p|}{M_1} + \frac{|\beta_p|}{2M_2} + \frac{|\gamma_p|}{4M_3} + \cdots$$

since then clearly $\lim_{p=\infty} \lambda_p = 0$, so that λ_p is in $\mathfrak{M}$ and also, by definition, will dominate the members of the sequence; in case III we may take λ_p so that

$$\lambda_p^2 = \frac{\alpha_p^2}{M_1} + \frac{\beta_p^2}{2M_2} + \frac{\gamma_p^2}{4M_3} + \cdots,$$

where $M_1, M_2, \cdots$ exceed the sum of the squares for the sequences $\alpha_p, \beta_p, \cdots$ respectively, since then clearly $\sum_{p=1}^{\infty} \lambda_p^2$ is a convergent series and also λ_p is a dominating function for each sequence.

Suppose now that any class $\mathfrak{M}$ of functions be given which possesses the property D. We may extend the class $\mathfrak{M}$ by adding to it all finite sums of constant multiples of functions of $\mathfrak{M}$. Since by D we may find a dominating function μ_p of $\mathfrak{M}$ for the functions in this sum, every function of this extended linear class $\mathfrak{M}_L$ does not exceed in absolute value a constant multiple of some function of $\mathfrak{M}$, for example $m\mu_p$, where μ_p is chosen as indicated and m is a sufficiently large constant. Let us now add to $\mathfrak{M}_L$ the limit functions ϕ_p of every convergent sequence of functions of $\mathfrak{M}_L$ which is relatively uniformly convergent as to a function λ_p of $\mathfrak{M}_L$; such a sequence may be written in the form of an infinite series of functions of $\mathfrak{M}_L$

$$\phi_p = \alpha_p + \beta_p + \cdots$$

since $\mathfrak{M}_L$ is linear. Inasmuch as λ_p does not exceed in absolute value some function of $\mathfrak{M}$, we may take the scale function λ_p to be in $\mathfrak{M}$. This extension of the class $\mathfrak{M}_L$ is called the $*$-extension of $\mathfrak{M}$ and is denoted by $\mathfrak{M}_*$. Since the remainder after n terms in the above series may be taken not to exceed $|\lambda_p|$ itself, by taking n sufficiently large, and since we can find a dominating function for each of these n terms and for λ_p which belongs to $\mathfrak{M}$, the same argument that was made above shows that ϕ_p does not exceed in absolute value some constant multiple of a function of $\mathfrak{M}$. Hence $\mathfrak{M}_*$ possesses the dominance property also, and the dominating function of any

sequence may be taken in $\mathfrak{M}$ (cf. Theorem, page 53); it is apparent that $\mathfrak{M}_*$ is linear.

A second extension $(\mathfrak{M}_*)_*$ *adds no new functions* (cf. Theorem III, page 52). For arrange the elements of the twice extended set as a double array of functions of $\mathfrak{W}$

$$\begin{aligned}\phi_p = \quad & (\alpha_p + \beta_p + \cdots) \\ & + (\alpha'_p + \beta'_p + \cdots) \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot,\end{aligned}$$

where the series in the successive rows converge relatively uniformly with respect to functions λ_p, λ'_p, $\cdots$ of $\mathfrak{M}$ and where the series of rows converges relatively uniformly with respect to a function $\bar{\lambda}_p$ of $\mathfrak{M}_*$. According to what has been said, the sequence $\bar{\lambda}_p$, λ_p, λ'_p, $\cdots$ possesses a single dominating function in $\mathfrak{M}$. All the scale functions may therefore be taken to be λ_p so that by taking a sufficiently large but finite number of elements of the double array we may obtain a function of $\mathfrak{M}_L$ which differs from ϕ_p by a quantity not exceeding $\epsilon\lambda_p$ in absolute value. Hence, we may replace the double array by a single array relatively uniformly convergent as to λ_p. This proves the statement to the effect that $\mathfrak{M}_*$ is closed under $*$-extension (cf. Theorem I, page 80).

By adding to $\mathfrak{M}$ its absolute value functions, one obtains $\mathfrak{M}_A$, and then by extending $\mathfrak{M}_A$ to be linear one obtains $(\mathfrak{M}_A)_L$; all functions obtained from $\mathfrak{M}$ by a succession of operations of this character form a class $\mathfrak{M}_{AL}$ with the properties *A*, *L*, *D*. The $*$-extension of $\mathfrak{M}_{AL}$ is called the $\sharp$-extension of $\mathfrak{M}$ and of course has the property *A* in addition to *L*, *C*, *D* for when any sequence of functions of $\mathfrak{M}_{AL}$ is relatively uniformly convergent as to a function of $\mathfrak{M}_{AL}$ the absolute value sequence will lie in $\mathfrak{M}$ and have the same convergence property. Thus we have

$$(\mathfrak{M}_\sharp)_\sharp = [(\mathfrak{M}_\sharp)_{AL}]_* = (\mathfrak{M}_\sharp)_* = [(\mathfrak{M}_{AL})_*]_* = (\mathfrak{M}_{AL})_* = \mathfrak{M}_\sharp;$$

hence $\mathfrak{M}_\sharp$ is closed under $\sharp$-extension (cf. Theorem II, page 80).

In the logical excursus of §§28–42 of Part I, Professor Moore considers the characteristic logical schemes suggested by the ideas of 'extension' and 'closure.' Thus for the subclasses $\mathfrak{N}$ containing $\mathfrak{M}$ of a fundamental class $\overline{\mathfrak{M}}$ the extension of $\mathfrak{M}$

as to P is the greatest common subclass of the classes containing $\mathfrak{M}$ and having the property P.

Part I concludes with a proof of the complete independence of the properties A, L, C, D: that is, it is proved that all the $2^4 = 16$ conceivable combinations of these properties and their negatives are consistent.

In Part II are treated the composition of classes $\mathfrak{M}$ and $\mathfrak{M}'$ on the respective ranges p and p'. If both $\mathfrak{M}$ and $\mathfrak{M}'$ possess the property D, the product class $\mathfrak{M}\mathfrak{M}'$ on the product range pp' obtained by multiplying any function ϕ_p of $\mathfrak{M}$ by any function $\phi'_{p'}$ of $\mathfrak{M}'$, has also the property D: for from any sequence $\alpha_p\alpha'_{p'}$, $\beta_p\beta'_{p'}$, $\cdots$ we deduce dominating functions λ_p for the sequence $\alpha_p, \beta_p, \cdots$ and $\lambda'_{p'}$ for $\alpha'_{p'}, \beta'_{p'}, \cdots$, and thus obtain a dominating function $\lambda_p\lambda'_{p'}$ in $\mathfrak{M}\mathfrak{M}'$ for the given sequence.

We may now obtain $(\mathfrak{M}\mathfrak{M}')_*$ which is readily shown to be the same as $(\mathfrak{M}_*\mathfrak{M}')_*$, $(\mathfrak{M}\mathfrak{M}_*')_*$ or $(\mathfrak{M}_*\mathfrak{M}_*')_*$ (cf. Theorem I, page 95). In fact the classes $\mathfrak{M}_*\mathfrak{M}'$ and $\mathfrak{M}\mathfrak{M}_*'$ form part of $(\mathfrak{M}\mathfrak{M}')_*$ and contain $\mathfrak{M}\mathfrak{M}'$. Hence the statement must be true for the first two of the three classes since $(\mathfrak{M}\mathfrak{M}')_*$ is the same as its $*$-extension. By a second extension as to p' one concludes that $\mathfrak{M}_*\mathfrak{M}_*'$ belongs to $(\mathfrak{M}\mathfrak{M}')_*$ and at the same time contains of course $\mathfrak{M}\mathfrak{M}'$. Hence the statement is true for the third class. Similar results may now be given at once for the composition of more than two classes (cf. Theorems II, III, IV, pages 95, 96).

In the special case where $\mathfrak{M}$ and $\mathfrak{M}'$ are the classes of all continuous functions of p and p' respectively, the class $(\mathfrak{M}\mathfrak{M}')_*$ is the class of all continuous functions of the two variables. This well known fact serves to orient one toward the remainder of Part II. The continuity in p, p' of a function $\phi_{pp'}$ of this class $(\mathfrak{M}\mathfrak{M}')_*$ is equivalent to the following two conditions (1) that $\phi_{pp'}$ is uniformly continuous in p, *uniformly over the range* p'; (2) $\phi_{pp'}$ is continuous in p' for every p. Professor Moore treats properties B of $\mathfrak{M}$ such that for every $\mathfrak{M}'$ with properties L, C, D every function of the class $(\mathfrak{M}\mathfrak{M}')_*$ is in $\mathfrak{M}$ for every p' and has the property B uniformly, and is in $\mathfrak{M}'$ for every p (cf. §§ 57–65). Of course the precise kind of uniformity needs to be stated in each case. The fact that $(\mathfrak{M}\mathfrak{M}')_*$ in the particular case is the class of all continuous functions of pp' suggests a consideration of the class $\mathfrak{F}$ of *all* functions of pp' belonging to the class $\mathfrak{M}$ and having a property B uniformly for every p' and belonging to the class $\mathfrak{M}'$ for every p (cf. Theorem, page 109, Theorem I, page 110).

From here on, while the range p' is kept general, the range p is taken to have a certain additional property Δ closely allied to the limit property of Fréchet, but of a *non-metric* character.* In the application of the General Analysis it is found necessary to have such properties, in order to secure a satisfactory characterization of functions of $(\mathfrak{M}\mathfrak{M}')_*$ on the range pp'.†

Professor Moore has found it possible to define the analog of the class of all continuous functions over ranges of this general description including the types presented in I–IV. The property of these ranges which is generalized is that of the denumerability in I, II, III and that of sequential halving of the interval in IV.

The generalization is obtained as follows: A development Δ of the range p is a sequence $\Delta^m (m = 1, 2, \cdots)$ of systems of subclasses $\Delta^{ml} (l = 1, 2, \cdots, l_m)$ of elements of the range, for a particular m giving the mth stage Δ^m of the development Δ. A function ϕ_p dominated by a function of $\mathfrak{M}$ is said to have the property K_{12} relative to $\mathfrak{M}$ and Δ (generalization of convergence and continuity) in case the difference between the values of ϕ_p at a fixed point p^{ml} and a variable point p in any one class Δ^{ml} of the mth stage tends to zero, as m increases, relatively uniformly to some function of $\mathfrak{M}$, and at the same time ϕ_p itself tends to zero in the same manner for points p in no class Δ^{ml} of the mth stage. It is then easy to demonstrate that all functions of $\mathfrak{M}_*$ possess the property K_{12} if those of $\mathfrak{M}$ do, and also that in case $\mathfrak{M}'$ is a second class of functions having the property K_{12} with respect to $\mathfrak{M}'$ and Δ', the set $\mathfrak{M}\mathfrak{M}'$ possesses the property K_{12} with respect to the composite development $\Delta\Delta'$ (cf. Theorem II, page 135).

Suppose now that we have a set of functions of $\mathfrak{M}$, say δ_p^{ml}, one for each subset of the development Δ of the range, such that (1) the sum of the absolute value of the functions $\delta_p^{ml}\phi_{p^{ml}}$ (p^{ml} a representative point of Δ^{ml} and ϕ_p with the property K_{12}) whose corresponding Δ^{ml} *does not* contain p becomes small relatively uniformly with respect to some function ϕ_{0p} of $\mathfrak{M}$ as m increases, (2) the sum of those δ_p^{ml} whose corresponding Δ^{ml} *does* contain p, as well as the sum of their absolute values, tends toward 1 uniformly as m increases. This set of functions

* Dr. T. H. Hildebrandt deals with the matter in his Chicago thesis of 1910.

† See the article by Professor Moore in the *Atti* of the Rome International Congress, 1908, vol. 2, pp. 98–114.

have clearly the property that any such function ϕ_p may be written as

$$\lim_{m=\infty} \sum_m \phi_{p=l} \delta_p^{ml}.$$

In fact we have

$$|\phi_p - \sum_m \phi_{p=l} \delta_p^{ml}| \leqq |\phi_p - \sum_{m_1} \phi_{p=l} \delta_p^{ml}| + |\sum_{m_2} \phi_{p=l} \delta_p^{ml}|,$$

where the first and second summation are extended respectively to the functions δ_p^{ml} whose Δ^{ml} does and does not contain p. But by (1) the second term on the right is not greater than $\epsilon |\phi_{0p}|$, where ϵ is arbitrarily small for m large, and ϕ_{0p} is a function of $\mathfrak{M}$. Also the first term is not greater than

$$|\phi_p - \Sigma_1 \phi_p \delta_p^{ml}| + |\Sigma_1 (\phi_p - \phi_{p=l}) \delta_p^{ml}|.$$

By (2) the first member of this sum does not exceed $\epsilon|\phi_p|$ for m large if there are any terms in Σ_1, and by K_{12} does not exceed $\epsilon|\phi_{1p}|$ in the contrary case (p not in any Δ^{ml}). The second term does not exceed $\epsilon|\phi_{2p}|$ by K_{12} and (2) or else is zero. Hence any function ϕ_p with the property K_{12} can be represented as stated and the convergence is relatively uniform with respect to some function of $\mathfrak{M}$ (namely with respect to any function which dominates ϕ_p, ϕ_{0p}, ϕ_{1p}, ϕ_{2p}), since $\mathfrak{M}$ possesses the property D. *It follows that* ϕ_p *belongs* to $\mathfrak{M}_*$ (cf. Theorem I, page 140).

The simplest illustration of the above is afforded by II, III when

$$\delta_p^{ml} = \begin{cases} 0, p \neq l \\ 1, p = l \end{cases} \quad \text{and} \quad l = 1, 2, \cdots, m.$$

In case IV a developmental system is well known also.

The composition theory of developments Δ and of developmental systems δ_p^{ml} is now finally considered; the functions of the composite developmental systems for any stage m are given by the products of the developmental functions of each of the classes for the same stage.

The notational scheme of the General Analysis makes possible a very abbreviated statement of the results. For example the last theorem dealt with above is written

$$\mathfrak{M}^{DK_{12}\Delta} \cdot \supset \cdot \mathfrak{M}_* = [\text{all } \phi^{K_{12}\mathfrak{M}}].$$

This affirms that if $\mathfrak{M}$ possesses the properties: D, the domin-

ance property; K_{12}, that every function of $\mathfrak{M}$ has the property K_{12} [relative to $\mathfrak{M}$ and some development Δ]; Δ, that $\mathfrak{M}$ possesses a developmental system of functions [relative to this same development Δ]; it follows that $\mathfrak{M}_*$ is the class of all functions having the property K_{12} relative to $\mathfrak{M}$ [and to Δ]. This example shows two of the principal features of the notation, namely the employment of superscripts to indicate properties, and the use of the Peano symbols.

There can be no doubt that the principal mathematical results of these lectures are of a simple character, admitting of very brief proof, and the reviewer has tried to bring out this fact. But Professor Moore has broken up his treatment into its component abstract parts, and at the same time has employed a sufficiently extensive technical notation to distinguish numerous special cases by their abstract properties; in this way a complex mathematical situation has arisen. The reason which led Professor Moore to adopt this form of treatment lies of course in the indisputable fact that the whole of mathematics needs to be presented from a standpoint which recognizes common elements of thoughts in diverse fields. It is Professor Moore who has most consistently advanced this important thesis.

The following list of *errata* has been forwarded by Professor Moore to the reviewer:

P. 128. A definition (6) is needed for use in § 74 (2). After lines 5 and 9 respectively *insert:*

$$(6)\quad \begin{matrix} K'_{p'_1 p'_2 m} \cdot p'' \cdot \supset \cdot K_{p'_1 p'' p'_2 p'' m}; \\ p' \cdot K''_{p'_1 p'_2 m} \cdot \supset \cdot K_{p' p'_1 p' p'_2 m}. \end{matrix}$$

$$(6)\ (K'_2, K_{12})^6, (K''_2, K_{12})^6, (K'_2, K''_2, K_{12})^6;$$

P. 129, ll. 3–5. *For* final superscript ^{3} *read* ^{5}.

P. 143, ll. 16–26. From the equation

$$\theta_{mp} - \phi_p = \sum_h {}^* \phi_{r^{mh}} \delta_p^{mh} - \sum_g (\phi_p - \phi_{r^{mg}}) \delta_p^{mg} + \phi_p (\sum_g \delta_p^{mg} - 1),$$

by suitable use of (19, 12, 15, 14, 16) we obtain the relation

$$(21)\ p^m \cdot (m \geqq m_0, m_{3e}, m_{2e}, m_{4e}) \cdot \supset \cdot$$
$$A(\theta_{mp} - \phi_p) \leqq e(A\mu_{3p} + 4A\mu_{2p} + aA\mu_{0p}).$$

P. 147, l. 15. *After* in fact, *insert*

in case $\mathfrak{M}'^{D_1K_2'}$ and $\mathfrak{M}''^{D_1K_2''}$,

P. 147, ll. 20–21. Insert the respective conditions

$$D_1K_2',\ D_1K_2''\,;\ K_2',\ K_2'',$$

on the classes $\mathfrak{M}'$, $\mathfrak{M}''$ in the hypotheses of the propositions of lines 20; 21.

P. 148, l. 12. *For* $D\Delta$ *read* $DK_2^{\Delta}\Delta$.

P. 148, l. 25. *For* $LCD\Delta$ *read* $LCDK_2^{\Delta}\Delta$.

II. *Projective Differential Geometry.*

In these lectures Professor Wilczynski presented an outline of some of the most important results of the projective differential geometry of curves and ruled surfaces, and at the same time gave an indication of the method of proof. A complete presentation of these results will be found in Professor Wilczynski's treatise on Projective Differential Geometry. For this reason the reviewer confines himself to a very brief synopsis of these readable lectures.

Projective differential geometry deals with the differential properties of geometrical configurations that are invariant under the projective group.

The curve in $(n-1)$ space presents itself as given by n homogeneous coordinates $y_1, \cdots, y_n$, each a function of the parameter x. Any other set of such coordinates in the same parameter will be given by any n linearly independent combinations of these, aside from a multiplicative transformation. Hence the homogeneous linear differential equation of the nth order, a fundamental set of whose solutions is $y_1, \cdots, y_n$ is the same as that given by any other projectively equivalent curve, or else is obtained from it by a multiplicative transformation of the dependent variable. If the factor is chosen so as to make the $(n-1)$th derivative disappear, the ratios of the remaining coefficients will then be projective invariants of a given curve with given parameter and are called the semi-invariants. The semi-covariants are similarly defined.

The absolute invariants and covariants are further invariant and covariant under a transformation of the parameter and are readily obtainable from the semi-invariants and covariants.

The geometrical interpretation of these functions leads to the projective differential geometry of curves, and is not given in the lectures by Professor Wilczynski. His contribution in this direction has been to make systematic use of the elegant analytic instrument afforded by the differential equation, and thus to obtain Halphen's results for the plane and space, and in addition like results in a general space.

The projective theory of the ruled surfaces in space whose generators are the lines joining corresponding points of two curves C_y and C_z with coordinates

$$y_1, y_2, y_3, y_4 \text{ and } z_1, z_2, z_3, z_4$$

is then treated. The most general set of coordinates (aside from a factor) is obtained by taking the same four linearly independent linear combinations of the y's and z's. Thus we are led to consider

$$\sum c_i y_i \text{ and } \sum c_i z_i.$$

These functions form the general solution of a pair of ordinary linear differential equations

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0.$$

Conversely any set of four linearly independent particular solutions y, z of such equations yield always projectively equivalent ruled surfaces. These equations are fundamental for the theory which has been created by Professor Wilczynski.

A change of the dependent variables

$$\bar{y} = \alpha(x)y + \beta(x)z, \quad \bar{z} = \gamma(x)y + \delta(x)z$$

changes C_y and C_z into any other pair of curves $\bar{C}_y$ and $\bar{C}_z$ on the same ruled surface, and a change of independent variable $\bar{x} = f(x)$ changes the parameter on the curves in any given way. The invariants and covariants under these transformations are now obtained analytically and their interpretation leads to the geometrical results.

Since the space dual of a ruled surface is a second ruled surface one may expect a second pair of adjoint equations to be associated with the given one. The invariants are the same for

the adjoint equations as for the given equations, except in sign. The condition that these two sets of equations are the same leads to the theorem that a ruled surface is projectively equivalent to its dual only in case it is a quadric.

The asymptotic lines on ruled surfaces are clearly projectively invariant. The condition that the curves C_y and C_z are asymptotic lines is simply $p_{12} = p_{21} = 0$ and in this case the integral curves of the given equations form the asymptotic lines. A simple consequence is the theorem of Paul Serret: the cross ratio formed by the intersection of a moving generator with four fixed asymptotic lines is constant.

The totality of the tangents to the asymptotic lines along a given generator g form one set of rulings of an osculating hyperboloid of the ruled surface. The totality of generators of the hyperboloids thus constructed form a congruence Γ. One of the covariants P gives a unique second generator g' of the hyperboloid H of the same set as g and thus a second ruled surface varying arbitrarily with the choice of independent variable of which it serves as the image. This ruled surface is the derivative ruled surface S' of S with respect to x.

The flecnode curves are two curves along which the tangents to the asymptotic lines have four points in common with the ruled surface. The flecnode points are determined by the equating to zero of a certain quadratic covariant C. If the flecnode points coincide, one of the invariants θ_4 vanishes. The developables of which the flecnode curves are the cuspidal edges form the focal surfaces of the congruence Γ. The condition that another invariant Δ vanishes is that the ruled surface belongs to a linear complex.

In the final lecture there are given first some theorems concerning the derivative surface of S. The lecturer then passes on to develop the notion of the two complex points (arising from another quadratic covariant) on each generator which with the flecnode points form a harmonic group. Lastly the theorems which state the extent to which the flecnode curves may be arbitrarily assigned are given.

III. *Theory of Boundary Value Problems.*

Professor Mason considers a functional equation $f = g + Sf$ for x on a range R, where S is a linear operator such that (1) $S\phi$ is continuous when ϕ is continuous, (2) the series

$$\phi + S\phi + S^2\phi + \cdots$$

converges uniformly in R, and (3) the result of the operation S on the series is the same as operating with S term by term. In this case S is suggestively termed a convergent operator and the infinite series above for $\phi = g$ is easily proved to form the unique continuous solution of the given equation.

As a first application, the existence theorem for ordinary linear differential equations of the second order is obtained. The method does not differ from the method of successive approximation, save in form.

The second application is to a linear partial differential equation in two independent variables of hyperbolic type. The standard reduction of linear partial differential equations of the second order to normal forms is first effected. In the hyperbolic case, this form is

$$\frac{\delta^2 u}{\delta x \delta y} = a\frac{\delta u}{\delta x} + b\frac{\delta u}{\delta y} + cu + f.$$

Professor Mason treats the following new boundary value problem: Given a rectangle $x_1 \leqq x \leqq x_2$ and $y_1 \leqq y \leqq y_2$ and upon it two curves $C_x : y = \phi(x)$ $(x_1 \leqq x \leqq x_2)$ and $C_y : x = \psi(y)$ $(y_1 \leqq y \leqq y_2)$ where ϕ, ψ, and the derivative of ψ are continuous, to determine a solution u of the above equation such that u and $\delta u/\delta y$ take assigned values $U(x)$ and $V(y)$ along C_x and C_y respectively. Professor Mason proves that a unique solution u exists. The critical portion of this demonstration is less satisfactorily presented than the remainder of the lectures. The operator S which appears in this case is shown to admit of a "majorant operator."

This boundary value problem contains most of those already considered for this type of equation, including the classical case in which the curves reduce to $x = x_0$ and $y = y_0$ respectively and that in which u and its normal derivative are given along one and the same monotonic curve.

The simplest equation of elliptic type, which is the potential equation in the plane, is next considered, and the method of Neumann is employed to show that a unique solution exists for a convex region bounded by a regular curve $x = \xi(t)$, $y = \eta(t)$ which takes given values along the boundary. In this case the operator S is

$$S(\phi) = \frac{1}{\pi}\int_0^t \left\{ [\phi(s) - \phi(t)] \frac{\delta}{\delta t} \arctan \frac{\eta(s) - \eta(t)}{\xi(s) - \xi(t)} \right\} dt,$$

which is also proved to be convergent. A short review of the more general methods which permit one to consider regions not convex is given.

The Green's function $G(x, y : \xi, \eta)$ for the potential equation is exhibited in the customary way by means of the result just stated. Then follows the interesting question of the existence of doubly periodic solutions u of $\Delta u = f(x, y)$, where $f(x, y)$ is doubly periodic in x, y with periods a, b. The real part of $\log \sigma(z - \xi) - \log \sigma(z - \eta)$, where σ is the Weierstrassian σ-function and $z = x + iy$, furnishes a solution of $\Delta u = 0$ analytic at all points of the period rectangle save for an infinity at $z = \xi$ and $z = \eta$ like $\pm \log r$ (r the distance from x, y to ξ, η) and furthermore is doubly periodic except for simple linear terms. By a slight modification destroying these terms the Green's function for the equation $\Delta u = 0$ and the boundary conditions $u(\alpha) = u(\alpha + a)$, $u(\beta) = u(\beta + b)$ are obtained. The necessary and sufficient condition for a doubly periodic solution is found by Professor Mason to be that the integral of f over the rectangle vanishes.* The proof depends on the properties of the Green's function.

It is next shown that a solution of

$$\Delta u + cu = f$$

taking assigned values on the boundary of a region R exists if R is taken sufficiently small. The operator S is obtained by means of the Green's function of $\Delta u = 0$.

Attention is then called by the lecturer to the recent work of Serge Bernstein, in which are proved simple criteria for the analytic or non-analytic nature of solutions of partial differential equations of the second order.

In conclusion, the lecturer considers the boundary value problem attaching to the equation and condition

$$y'' + \lambda A(x)y = 0, \quad y(a) = y(b) = 0, \quad (\lambda \text{ a parameter}),$$

as it arises from the consideration of the transverse vibrations of a stretched string. The method of the existence prpof depends on the solution of a certain minimum problem, and in order to

*See an article in the *Transactions of the American Mathematical Society*, vol. 6 (1905), p. 159.

carry through this proof the general solution of

$$y'' + \lambda A(x)y = f, \quad y(a) = y(b) = 0$$

is first derived. The minimum problem which is to be treated is to minimize $\int_a^b y'^2 dx$ under the condition that $\int_a^b Au^2 dx = 1$ and $\int_a^b y_i y dx = 0$ $(i = 1, 2, \cdots, n)$, where $y_1, \cdots, y_n$ are the solutions belonging to $\lambda_1, \cdots, \lambda_n$. This gives in order of increasing magnitude the positive values of λ; these exist in infinite number if ϕ is anywhere positive. Likewise a series of negative values of λ will be obtained if ϕ is anywhere negative.*

A formal expansion of an arbitrary function is then given by

$$f = \sum_{-\infty}^{+\infty} c_i y_i, \quad c_i = \pm \int_a^b f A_i y_i dx,$$

and Professor Mason states the theorem that this expansion holds if f vanishes at a and b, is continuous, and has a derivative continuous save at a finite number of points. The proof however contains an error.†

G. D. BIRKHOFF

SHORTER NOTICES.

Vorlesungen über Algebra. Von GUSTAV BAUER. Herausgegeben vom Mathematischen Verein München. 2te Auflage. Leipzig und Berlin, B. G. Teubner, 1910. vi + 366 pp.

THAT Professor Bauer's lectures, which were published in honor of his 80th birthday by the Mathematical Club of Munich in 1903, are destined to outlive their author by many years, seems to be evidenced by the fact that a second edition became necessary in 1910, seven years after the first edition and four years after Professor Bauer's death, which occurred on April 3, 1906.

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*See an article in the *Transactions*, vol. 8 (1907), p. 373.

† At bottom of p. 219 it is necessary to replace the multiplier 2 in the inequality

$$(\phi_{n,m})^2 \leqq 2 \sum_{i=n}^{m} c_i^2 \left(\int^x y_i' dx \right)^2$$

by a multiplier $m - n$. This error appears to destroy the force of the proof.

A fine spirit of veneration for the deceased teacher and friend breathes from the preface for this second edition, written by Professor Karl Doehlemann, who has now taken full charge of the book. It is clear that Professor Bauer, whose portrait appears opposite the title page, was much respected by those who worked under him and with him.

As the first edition of these lectures was quite fully reviewed by Professor Dickson on pages 257–260 of volume 10 of the BULLETIN, the task of the present reviewer may be limited to a discussion of the points of difference between the two editions. In Part I, devoted to the general properties of algebraic equations, the most important change consists in the substitution of Ulherr's proof of the fundamental theorem of algebra for the proof of Argand, which only showed that if $f(z) \neq 0$, a complex number h could be found, such that $|f(z+h)| < |f(z)|$. Ulherr's proof makes use of the fact, that when z describes a circle about the origin in the z-plane, $w = f(z)$ describes a curve in the w-plane which goes about the origin n times, if $f(z)$ is of degree n.

The notion of a domain of rationality, formerly taken up in the chapter on the Galois theory, is now introduced in Chapter IV, as are also the notions of reducibility and irreducibility of functions and of equations.

Part II, which treats the algebraic solution of equations, has undergone greater changes than the other parts. The chapter on the Galois theory of algebraic equations has been omitted and a new chapter has been inserted on Abel's theorems concerning the general representation of algebraic functions. This makes possible a more rigorous presentation than was formerly possible of the Abel-Wantzel theorem that equations of degree higher than the fourth are not in general solvable by radicals. In this new Chapter XV, which follows exactly the lines of the classical treatise of Serret, one feels a dissimilarity from the older chapters in the complete absence of illustrative examples, so abundant elsewhere. The book, as stated in the preface, does not take the ultra-modern point of view, and is intended for students who for the first time are making a systematic study of the science of algebra. In this chapter, where they are getting a first introduction to the more difficult parts of the subject, a few well chosen examples would help them to keep the general statements connected with the concrete situations of which they are the abstraction.

Part III, in which the numerical solution of algebraic equations is taken up, is left practically unchanged.

In Part IV, theory and application of determinants, the chapter on linear equations (Chapter XXVI) has been almost completely rewritten by Dr. Perron. The subject is now presented more systematically, by the reduction of the solution of a system of n linear non-homogeneous equations in n variables to the solution of a system of n linear homogeneous equations in $n + 1$ variables. By introducing the notion of rank of a matrix (formerly postponed to the last chapter in the book) the results are stated more compactly. Although the treatment of this subject has thereby become much clearer, it seems to the reviewer that still further progress towards unifying it can be made, as is shown for instance on pages 43–53 of Bôcher's Introduction to Higher Algebra.

Probably in order to remove the objection to the proof of the multiplication theorem as given in the first edition, raised, for instance, in Professor Dickson's review, the theorem is now introduced that a determinant is an irreducible function of its n^2 elements whereby the difficulty referred to is removed.

One regrets the scarcity of geometrical illustrations and interpretations. The numerous examples illustrating the methods presented make the book valuable for the student. It seems strange that the treatment of determinants should be left for the last section of the book; if presented earlier, the subject of elimination, which is fully treated and is now taken up in Part I and again in Part IV, could be discussed more compactly. The presentation is on the whole very clear; it goes step by step and leads gradually to a full understanding of the topics discussed, which furnish a good introduction to the study of algebra.

It is remarkable, that while the typographical errors of the first edition have been corrected, nearly as many new ones have found their way into the second edition.

ARNOLD DRESDEN.

Théorie mathématique des Assurances. Par P. J. RICHARD et E. PETIT. Paris, Doin, 1908. 396 pp.

THE series of books coming out under the general title Encyclopédie Scientifique seems not to be very well known in this country, yet the enterprise is highly ambitious and indus-

triously carried on under the editorship of Dr. Toulouse of the École des Hautes-Études. The handy little volumes are "in 18 jésus cartonnés," which is better understood in the original than in a translation, contain about 400 pages, and sell uniformly at 5 francs. The individual volumes would doubtless carry considerable weight by themselves; but their value and influence is much enhanced by the fact that each fits into its appropriate nook in the vast Encyclopédie Scientifique, which is divided into 40 sections and which will contain about 1000 volumes in all. The fact that the volumes are printed separately makes it possible to keep them revised and up-to-date; the fact that they belong to a series serves as a check on unnecessary repetition; the limited but not too limited size of the volumes allows each to give a good idea of its field without developing into a ponderous treatise. The sub-editors under Toulouse are Painlevé for the philosophy of science, Drach for mathematics and for mechanics, D'Ocagne for applied mathematics and for applied mechanics; and the supervisors in other fields are equally qualified for their tasks.

Richard and Petit, who write the volume on insurance, are actuaries and graduates of the École Polytechnique. They have produced a book in the spirit of the whole series, readable, yet not merely popular. The theory of insurance is developed in general and is applied to the discussion of numerous kinds of policies. The standard mortality tables are those compulsory in France; the types of policy treated in greatest detail are those most in vogue in France. This is but natural, and naturally the book will be most serviceable in France. But the foundation of the work is so carefully laid that it may be of value in any country. The authors do not overlook the fact that in several instances the insurance companies find it necessary for business reasons to adopt a procedure not wholly in accord with that which the theory would indicate as most rational. Unfortunately insurance has to contend with the psychology of the insured and insurable. This is nowise more evident than from the fact that the French government, which for forty years has maintained an insurance department guaranteed by the state, with smaller premiums than the private companies, and with no requirement for a medical examination, has written only about a million and a half dollars of insurance in the whole period! This cannot be attributed to partiality against the state; for the government old-age pensions or insurance are

a great success, the receipts for 1906 alone being over forty million dollars. The attitude of the authors throughout is that of practical as well as scientific actuaries.

E. B. WILSON.

NOTES.

PROFESSOR KLEIN'S Evanston Colloquium Lectures on Mathematics, which have for some years been out of print, have been republished by the AMERICAN MATHEMATICAL SOCIETY and are now on sale at the nominal price of seventy-five cents per copy, postage free. The new edition is printed from the original plates, with correction of a few misprints. A brief preface by Professor W. F. OSGOOD has been added. The volumes are bound like the original. Orders should be addressed to the American Mathematical Society, 501 West 116th Street, New York, N. Y.

THE April number (volume 12, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Biorthogonal systems of functions," by ANNA J. PELL; "Applications of biorthogonal systems of functions to the theory of integral equations," by ANNA J. PELL; "On the uniform convergence of the developments in Bessel functions," by C. N. MOORE; "Determination of the ordinary and modular linear groups," by H. H. MITCHELL; "General theory of linear difference equations," by G. D. BIRKHOFF.

THE April number (volume 33, number 2) of the *American Journal of Mathematics* contains: "On three-spreads satisfying four or more homogeneous linear partial differential equations of the second order," by C. H. SISAM; "Some properties of lines in space of four dimensions and their interpretation in the geometry of the circle in space of three dimensions," by C. L. MOORE; "On the geometry of line elements in the plane with reference to osculating circles," by G. F. GUNDELFINGER; "Binary modular groups and their invariants," by L. E. DICKSON; "The group of turns and slides and the geometry of turbines," by EDWARD KASNER.

AFTER the completion of the present volume, the *Annals of Mathematics* will be published under the auspices of Princeton University instead of Harvard University as heretofore. The

editorial staff will consist of Professors ORMOND STONE, of the University of Virginia, MAXIME BÔCHER, of Harvard University, and G. D. BIRKHOFF, L. P. EISENHART, E. SWIFT, OSWALD VEBLEN, and J. H. MACLAGAN-WEDDERBURN, of Princeton University. Correspondence concerning the new volume should be addressed to the *Annals of Mathematics*, Princeton University, Princeton, N. J.

AT the meeting of the London mathematical society held on March 9 the following papers were read: By G. B. MATHEWS, "On the reduction and classification of binary cubic forms which have a negative determinant"; by A. E. H. LOVE, "The theory of the transmission of earthquake waves"; by P. A. MACMAHON, "The theory of partitions."

THE next annual meeting of the Deutsche Mathematiker-Vereinigung will be held at Carlsruhe in affiliation with the eighty-third convention of the society of German naturalists and physicians during the week beginning September 24. Titles and abstracts of papers to be presented should be sent to the secretary, Professor A. Krazer, Westendstrasse 57, Carlsruhe, before May 15.

THE arrangements for the next International congress of mathematicians, to be held at Cambridge, England in August, 1912, are well under way and it is expected that the details will be announced shortly.

THE eminent mathematician, Professor GASTON DARBOUX, of the University of Paris, is about to complete his fiftieth year of service as a teacher in the system of public instruction of France. For more than twenty-five years he has been a member of the Academy of Sciences and for the past ten years he has been its permanent Secretary. It is proposed by a large international group of his mathematical co-workers, friends, and former pupils to commemorate this anniversary by presenting to Professor Darboux a gold medal bearing his portrait, and an appropriate address signed by the participants. All mathematicians are invited to share in rendering this honor to Professor Darboux. Copies of the medal, in reduced size, will be struck. Subscribers of twenty-five francs will receive a copy in bronze, subscribers of fifty francs a copy in silver. Subscriptions should be sent to Professor Cl. Guichard, secretary of the Faculté des Sciences.

THE rector and faculties of the University of Naples propose that a commemorative tablet to the late Professor ALFREDO CAPELLI be placed in the Sala Battaglini. Those who wish to participate in the movement are invited to send their contributions to the director of the mathematical seminar, Professor Ernesto Pascal, Viale Elena 24, Naples.

THE subject of the Adams prize for 1914 will be "The theory of radiation."

THE royal academy of sciences of Belgium announces the following prize problem for 1912: "To systematize and complete the investigations made in the calculus of variations since 1850." Competing memoirs should be written in French or Flemish and sent to the permanent secretary of the academy before August 1, 1912.

THE following courses in mathematics are announced for the year 1911–1912:

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Mathematical physics, three hours.—By Professor J. H. TANNER: Teachers' course, three hours.—By Professor J. I. HUTCHINSON: Elliptic functions, three hours.—By Professor V. SNYDER: Projective geometry, three hours.—By Professor F. R. SHARPE: Mechanics, three hours.—By Professor W. B. CARVER: Theory of numbers, three hours (first term); Conjugate coordinates, three hours (second term).—By Dr. D. C. GILLESPIE: Theory of functions of a real variable, three hours.—By Dr. C. F. CRAIG: Algebraic curves, three hours.—By Dr. F. W. OWENS: Differential equations, two hours.—By Dr. J. V. MCKELVEY: Analytic geometry, three hours.—By Dr. L. L. SILVERMAN: Infinite series, three hours (first term); Algebra, three hours (second term).—By Dr. W. A. HURWITZ: Differential equations of mathematical physics, three hours.—By Dr. E. J. MILES: Advanced calculus, three hours.

PRINCETON UNIVERSITY.—By Professor H. B. FINE: Theory of elimination, three hours (first term).—By Professor H. D. THOMPSON: Infinitesimal geometry, three hours; Coordinate geometry, three hours.—By Professor L. P. EISENHART: Mechanics, three hours; Partial differential equations, three hours (first term); Vector analysis, three hours (second term). By Professor O. VEBLEN: Projective geometry, three hours;

Theory of functions of real variables, three hours.—By Professor G. D. BIRKHOFF: Analysis, three hours; Linear differential equations, three hours.—By Professor W. GILLESPIE: Theory of substitutions, three hours (first term).—By Professor J. G. HUN: Analytic projective geometry, three hours (second term).—By Professor E. SWIFT: Differential equations, three hours; Calculus of variations, three hours (second term).—By Professor J. H. MCL. WEDDERBURN: Theory of functions of a complex variable, three hours.

THE following courses in the German universities will be given during the present summer semester:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Integral calculus, with exercises, six hours; Selected chapters of elliptic functions, two hours; Calculus of variations, four hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of algebraic equations, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of functions, II, four hours; Theory of curvature of curves and surfaces, four hours; Seminar, two hours. — By Professor G. HETTNER: Introduction to the theory of determinants, two hours.—By Professor J. KNOBLAUCH: Analytic geometry, four hours; Theory of space curves, II, one hour; Theory of surfaces, II, four hours.—By Professor R. LEHMANN-FILHÉS: Differential calculus, four hours. — By Professor I. SCHUR: Ordinary differential equations, four hours; Introduction to the theory of functions, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Introduction to differential geometry, four hours; Seminar, two hours. — By Professor F. LONDON: Calculus, with exercises, four hours; Axonometry and perspective, two hours; Seminar, two hours. — By Professor F. HAUSDORFF: Theory of differential equations, four hours. — By Dr. J. O. MÜLLER: Calculus of variations, three hours. — By Dr. W. BLASCHKE: Algebraic equations, four hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Introduction to differential and integral calculus, four hours; Seminar, two hours. — By Professor D. HILBERT: Mechanics of the continuum, four hours; Seminar, two hours. — By Professor E. LANDAU: Elements of the theory of numbers, four hours; Seminar, two hours.—By Professor C. RUNGE: Num-

erical calculation, six hours; Photogrammetry, three hours.— By Professor F. BERNSTEIN: Theory of probabilities, three hours; Calculus of insurance, with exercises, three hours. — By Dr. O. TOEPLITZ: Theory of functions of a real variable, two hours. — By Dr. H. WEYL: Analytic geometry, four hours. — By Dr. A. HAAR: Differential equations, four hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Projective geometry, two hours; Plane analytic geometry, four hours; Conic sections and quadric surfaces from the standpoint of invariants, two hours; Seminar, two hours.—By Professor O. HÖLDER: Algebraic equations, two hours; General theory of functions of a complex variable, four hours; Seminar, two hours.—By Professor G. HERGLOTZ: Selected chapters of the calculus, one hour; Boundary values in the theory of potential, two hours; Seminar, two hours.—By Professor P. KOEBE: Ordinary differential equations, with exercises, five hours; Algebraic analysis, two hours.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Analytic geometry of space, five hours; Elliptic functions, five hours; Seminar, two hours.—By Professor A. VOSS: Introduction to the theory of partial differential equations, four hours; Introduction to the theory of invariants, three hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Integral calculus, five hours; Geometric applications of differential calculus, two hours.—By Professor H. BRUNN: Elements of higher mathematics, three hours.—By Professor K. DOEHLEMANN: Descriptive geometry, with exercises, six hours; Theory of algebraic constructions, with exercises, five hours.—By Dr. F. HARTOGS: Algebra, II, four hours; Algebraic analysis, five hours.—By Professor L. v. SEELINGER: Theory of probabilities and least squares, four hours. — By Professor A. SOMMERFELD: Mechanics of continua with applications, five hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Theory of numbers, two hours; Seminar, two hours. By Professor F. SCHUR: General theory of curves and surfaces, four hours; Introduction to vector analysis, two hours; Seminar, two hours.—By Professor J. WELLSTEIN: Integral equations, three hours; Theory of invariants, three hours; Proseminar, two hours.—By Professor R. E. v. MISES: Descriptive geom-

etry, four hours; with exercises, four hours; Theory of probabilities, two hours; Seminar, two hours.—By Professor P. EPSTEIN: Foundations of analysis, two hours.—By Professor M. SIMON: History of mathematics during the renaissance, two hours.

THE number of students of mathematics in the Prussian universities has increased from 1,440 in 1907–08 to 2,040 in 1910–11.

PROFESSOR TH. VAHLEN, of the University of Greifswald, has been promoted to a full professorship of mathematics.

DR. P. J. HEADWOOD has been appointed professor of mathematics at Durham University.

MR. R. D. CARMICHAEL, of Princeton University, has been appointed assistant professor of mathematics at Indiana University.

PROFESSOR J. ROSANES, of the University of Breslau, has retired from active service.

PROFESSOR F. A. SHERMAN, of Dartmouth College, will retire from active service at the close of the present academic year.

MR. A. R. MAXSON, for six years instructor in mathematics in Columbia University, died April 13, in his thirtieth year. He had been a member of the American Mathematical Society since 1905.

Recent catalogues of second hand mathematical books: A. Hermann et Fils, Paris, 6 rue de la Sorbonne, mathematics and sciences, about 150 titles in mathematics. — K. F. Koehler, Leipzig, Kurprinzstrasse 6, catalogue 586, mathematics, about 1200 titles. — G. E. Stechert, New York, 151 West 25th Street, catalogue 28, exact sciences, about 400 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

COFFIN (J. G.). Vector analysis. An introduction to vector-methods and their various applications to physics and mathematics. New York, Wiley, 1911. 12mo. 19+248 pp. Cloth. $2.50

FABRY (E.). Théorie des séries à termes constants. Applications aux calculs numériques. Paris, Hermann, 1911. 8vo. 203 pp. Fr. 6.50

——. Traité de mathématiques générales, à l'usage des chimistes, physiciens, ingénieurs. Avec une préface de G. Darboux. 2e édition, revue et augmentée. Paris, Hermann, 1911. 8vo. 10+462 pp. Fr. 9.00

FISCHER (P. B.). Koordinatensysteme. (Sammlung Göschen, 507.) Leipzig, Göschen, 1911. 8vo. 125 pp. Cloth. M. 0.80

FRÉCHET (M.). Les fonctions d'une infinité de variables. Paris, Imprimerie nationale, 1910. 8vo. 8 pp.

GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Unter Mitwirkung von J. Lüroth, E. Study, J. Grassmann, H. Grassmann d. J., G. Scheffers herausgegeben von F. Engel. 3ter Band, 1ter Teil. Leipzig, Teubner, 1911. 8vo. 5+353 pp. M. 18.00

HACK (F.). Wahrscheinlichkeitsrechnung. (Sammlung Göschen, 508.) Leipzig, Göschen, 1911. 8vo. 123 pp. Cloth. M. 0.80

HAMEL (A.). Petit traité d'analyse accompagné d'exercises pratiques. 4e édition, entièrement refondue, conforme à l'arrêté du 25 juillet 1910. Paris, Hatier. 18mo. 104 pp.

JUNKER (F.). Repetitorium und Aufgabensammlung zur Differentialrechnung. (Sammlung Göschen, 146.) 3te, verbesserte Auflage. Leipzig, Göschen, 1911. 8vo. 129 pp. Cloth. M. 0.80

KLEIN (F.). The Evanston Colloquium lectures on mathematics, delivered at Northwestern University Aug. 28 to Sept. 9, 1893. Reported by Alexander Ziwet. 2nd edition. New York, American Mathematical Society, 1911. 8vo. 11 + 109 pp. Cloth. $0.75

LEVY (A.). See SOMMER (J.).

LORIA (G.). Spezielle algebraische und transzendente ebene Kurven. Deutsch von F. Schütte. 2te Auflage. 2ter Band: Die transzendenten und die abgeleiteten Kurven. (Teubner's Sammlung, Band V, 2.) Leipzig, Teubner, 1911. 8vo. 8+384 pp. Cloth. M. 14.00

MEISER (W.). Lösungen zu Aufgaben aus der algebraischen Analysis. Nürnberg, Korn, 1910. 8vo. 3+66 pp. M. 2.00

METTLER (H.). Graphische Rechnungsmethoden im Dienste der Naturwissenschaften. Zürich, 1910. 71 pp. Cloth. M. 2.00

MINKOWSKI (H.). Gesammelte Abhandlungen. Unter Mitwirkung von A. Speiser und H. Weyl herausgegeben von D. Hilbert. 1ter Band. Leipzig, Teubner, 1911.

NAKAGAWA (S.). Miscellen aus dem Gebiete der hyperbolischen Geometrie. Tokyo, Journ. Coll. Sc., 1910. 4to. 68 pp. M. 2.40

PACI (E.) Sulla integrazione di un'equazione di Riccati. Palermo, Virzi, 1910. 8vo. 35 pp.

RABICH (O.). Die Kegelschnitte im Lehrplane der Seminarien. Eisenach, 1910. 8vo. 24 pp.

SCHÜTTE (F.). See LORIA (G.).

SOMMER (J.). Introduction à la théorie des nombres algébriques. Edition française revue et augmentée. Traduit de l'allemande par A. Levy, avec préface de J. Hadamard. Paris, Hermann, 1911. 8vo. 10+376 pp. Fr. 15.00

VINOT (J.). Récréations mathématiques. Questions curieuses et utiles extraites des auteurs anciens et modernes. 6e édition. Paris, Larousse, 1911. 8vo. 215 pp.

WIPPER (J.). 46 Beweise des pythagoräischen Lehrsatzes nebst kurzen biographischen Mitteilungen. 2te Auflage. Berlin, Barsdorf, 1911. 8vo. 51 pp. M. 1.50

II. ELEMENTARY MATHEMATICS.

ABHANDLUNGEN über die Reform des mathematischen Unterrichts in Ungarn. Nach dem ungarischen Original unter Mitwirkung von M. Balog und J. Rados deutsch herausgegeben von E. Beke und S. Mikola. Leipzig, Teubner, 1911. 8vo. 6+160 pp. M. 4.00

BARLOW'S tables of squares, cubes, square roots, cube roots, reciprocals of all integer numbers up to 10,000. London, Spon. 8vo. 208 pp. 4s.

BASSANI (A.). See LAZZERI (G.).

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik für höhere Mädchen-Bildungsanstalten. Nach modernen Grundsätzen bearbeitet. 1ter Teil. 2te Auflage. Leipzig, Teubner, 1911. 8vo. 8+348 pp. Cloth. M. 3.00

BIELER (A.). Lehr- und Uebungsbuch der Raumlehre für Knaben-Mittelschulen. 2te, nach den Lehrplänen von 1910 umgearbeitete Auflage. Leipzig, Teubner, 1911. 8vo. 7+176 pp. Cloth. M. 2.00

BORGOGNO (G.). Compendio di aritmetica, geometria e di sistema metrico decimale. 4ta edizione, riveduta ed accresciuta. Torino, Paravia, 1910. 16mo. 122 pp. L. 0.80

COLAW (J. M.). Elementary algebra. Richmond, Va., Johnson, 1911. 12mo. 368 pp. Half leather. $1.00

DOUGLAS'S two-color logs, anti-logs and mathematical tables. Prepared to meet the requirements of the leaving certificate examinations. London, Simpkin, 1911. 8vo. Sewed. 4d.

FÄRBER (C.). Arithmetik. (Die Grundlehren der Arithmetik und Algebra, bearbeitet von E. Netto und C. Färber. 1ter Band.) Leipzig, Teubner, 1911. 8vo. 15+410 pp. Cloth. M. 9.00

FREUD (P.). Die mathematischen Schulbücher an den Mittelschulen und verwandten Anstalten. (Berichte über den mathematischen Unterricht in Oesterreich, 6tes Heft.) Wien, Hölder, 1910. 8vo. 3+53 pp. M. 1.20

FROUMENTY (M.). See PHILIPPE (P.).

GAJDECZKA (J.). Prüfungsfragen aus der Mathematik. 3te, neubearbeitete Auflage der "Maturitätsprüfungsfragen." Wien, Deuticke, 1910. 8vo. 3+84 pp. M. 1.40

GAMBIOLI (D.). Geometria elementare. Roma, Bertero, 1910. 8vo. 8+461 pp. L. 3.25

GODFREY (C.) and SIDDONS (A. W.). Elementary geometry, practical and theoretical, together with solid geometry. London, Cambridge University Press, 1911. 8vo. 134 pp. 4s. 6d.

——. Solid Geometry. London, Cambridge University Press, 1911. 8vo. 120 pp. Sewed. 1s. 6d.

GÖTTING (E.). See BEHRENDSEN (O.).

HERTING (G.). I: Von Strecke, Quadrat und Würfel zum bestimmten Integral. Leipzig, Teubner, 1910. 8vo. 4+135 pp. Cloth. M. 2.80

HOČEVAR (F.). Lehr- und Uebungsbuch der Geometrie für Gymnasien und Realgymnasien. 7te Auflage. Wien, Tempsky, 1911. Kr. 2.60

HOWE (G.). Mathematics for the practical man, explaining simply and quickly all the elements of algebra, geometry, trigonometry, logarithms, coordinate geometry, calculus. New York, Van Nostrand, 1911. 12mo. 5+143 pp. $1.25

HUPE (A.). See MÜLLER (H.).

KIND (H. W.). The model practical mensuration. Based upon elementary geometric constructions arranged to suit the present gradation of school work. New York, Hinds, 1911. 12mo. 89 pp. $0.50

LAAGER (F.). Vereinfachter Lehrgang der Elemente der Trigonometrie. Zürich, Speidel, 1911. 8vo. 40 pp. M. 1.20

LANGEL (W.). See NIELSEN (C.).

LAZZERI (G.) und BASSANI (A.). Elemente der Geometrie. Unter Verschmelzung von ebener und räumlicher Geometrie. Aus dem Italienischen von P. Treutlein übersetzt. Leipzig, Teubner, 1911. 8vo. 16+491 pp. Cloth. M. 14.00

LUCHERINI (R.). Fascicoli di calcolo per le scuole elementari. Nuova edizione, corretta e modificata. Firenze, Bemporad, 1910. 16mo. 136 pp. L. 0.65

MATHEMATICAL questions and solutions. Volume 18. New series. London, Hodgson, 1911. 8vo. 6s. 6d.

MATRICULATION mathematics papers. Being the papers in elementary mathematics, set at the matriculation examination of the University of London, from June, 1903 to January, 1911. (University tutorial series.) London, Clive, 1911. 8vo. Sewed. 1s. 6d.

MATRICULATION model answers, mathematics. London University matriculation papers in mathematics from September, 1907 to January, 1911. (University tutorial series.) London, Clive, 1911. 8vo. Sewed. 2s.

MILLIS (J. F.). See STONE (J. C.).

MÖNKEMEYER (K.) und RÜSEWALD (K.). Lehr- und Uebungsbuch der Mathematik für höhere Mädchenschulen. Auf Grund der neuen Lehrpläne bearbeitet. 1tes, 2tes und 3tes Heft. Leipzig, Quelle und Meyer, 1911. 8vo. 45, 66 and 63 pp., respectively. M. 2.40

MORITZ (R. E.). College mathematics notebook for classes in advanced algebra, trigonometry, analytic geometry and calculus. Boston, Ginn, 1911. 4to. $0.80

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe B, unter Mitwirkung von A. Hupe. 2ter Teil: Die Oberstufe. 1te Abteilung: Planimetrie, Algebra, Trigonometrie und Stereometrie. 4te Auflage. Leipzig, Teubner, 1910. 8vo. 12+308 pp. Cloth. M. 3.00

MÜLLER (O.). Tavole di logaritmi con cinque decimali. 11a edizione, aumentata delle tavole dei logaritmi d'addizione e sottrazione, per cura di M. Rajna. (Manuali Hoepli, XLI.) Milano, Hoepli, 1911. 16mo. 36+191 pp. L. 1.50

NEVEU (H.) Cours d'algèbre théorique et pratique, suivi de notions de trigonométrie. 5e édition, conforme aux programmes de 26 Juillet 1909. Paris, Masson, 1911. 16mo. 6+480 pp. Fr. 3.00

NEVEU (H.) et BELLENGER (H.). Cours de géometrie théorique et pratique. 3e année. Paris, Masson, 1911. 16mo. 403 pp. Fr. 3.00

NIELSEN (C.) und LANGEL (W.). Planimetrie und Stereometrie für Landwirtschaftsschulen. Berlin, Parey, 1911. 8vo. 7+159 pp. Cloth. M. 2.50

PHILIPPE (P.) et FROUMENTY (M.). Cours de géométrie, à l'usage des écoles pratiques de commerce et d'industrie. Tome 1er: 1re année. Tome 2: 2e année. Paris, Dunod et Pinat, 1910. 8vo. 244+245 pp. Fr. 8.00

PINCHERLE (S.). Esercizi sull'algebra elementare. 2a edizione. (Manuali Hoepli, serie scientifica, n° 206.) Milano, Hoepli, 1911. 24mo. 6+136 pp. L. 1.50

ROBINSON (A. D.). The arithmetic help. For pupils, teachers, business men. Akron, O., Werner, 1911. 8vo. 515 pp. $1.00

RÜSEWALD (K.). See MÖNKEMEYER (K.).

SCOCCIANTI (F.). Compendio di trigonometria piana. Viterbo, Agnesotti, 1910. 8vo. 27 pp. L. 0.30

SPERA (S.). Elementi di algebra, per le scuole tecniche. Milano, Operai, 1911. 16mo. 94pp. L. 1.25

STONE (J. C.) and MILLIS (J. F.). Plane and solid geometry. Boston, Sanborn, 1911. 8vo. 400 pp. $1.30

——. Plane geometry. Boston, Sanborn, 1911. 8vo. 260 pp. $0.85

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III. APPLIED MATHEMATICS.

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